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BASIC INTEGRATION

By F. H. JACKSON (*Eastbourne*)

[Received 9 March 1948; in revised form 3 October 1949]

1. Introduction

In a former number of this Journal* I gave some examples of an operation which I named 'q-integration'. In this present paper the operation is considered more generally and precisely. The earlier sections of the paper deal with the general theory of basic integration and its relation to Riemann's definition of an integral. The later sections contain examples covering a wider range than those given in the previous paper and are chiefly connected with basic integrals analogous to

$$\int_0^\infty \Phi(-x)x^{s-1}\,dx = \Gamma(s)\phi(-s),$$

where
$$\Phi(x) = \phi(0)+\frac{x}{1!}\phi(1)+\frac{x^2}{2!}\phi(2)+\dots. \tag{1}$$

This integral formally due to Ramanujan is discussed with precision by G. H. Hardy.†

In the analysis of basic functions much use is made of the finite difference operation $\mathfrak{D}\Phi(x) = \{\Phi(x)-\Phi(qx)\}/x$ but little or no use has been made hitherto of the inverse operation. This is somewhat remarkable considering that integration rather than differentiation is the powerful tool in mathematical analysis. In this present paper I write

$$(1-q)\mathfrak{D}^{-1}\Phi(x) \equiv \mathsf{S}\,\Phi(x)\,d(qx),$$

and name this inverse operation *basic integration*, showing that in the limit, as $q \to 1$, the basic integral reduces to ordinary integration. Among the examples in the later sections of the paper a very simple basic integral gives an identity which contains as special cases numerous identities given by Jacobi in *Fundamenta Nova*, which are of interest in theta-function theory and theory of numbers.

Notation. I use two bases p, q, where

$$p > 1 \quad (p = 1+\epsilon), \qquad q < 1 \quad (q = 1-\epsilon),$$

and write

$$[a] \equiv 1-q^a, \qquad \{a\} \equiv p^a-1,$$
$$[a]_n = [a][a+1]\dots[a+n-1], \qquad [n]! = [1]_n.$$

* F. H. Jackson, *Quart. J. of Math.* (1910), 193–203.

† G. H. Hardy, *Ramanujan* (Cambridge, 1940), Ch. XI.

Here p^a, q^a denote principal values only, and all letters p, q, a, b, x,... denote real numbers. The special brackets prevent confusion with (a) which denotes an ordinary number a.

2. Notation for symbolic operators $\mathfrak{D}$, $\mathfrak{D}^{-1}$

I use the symbolic operators $\mathfrak{D}_p$, $\mathfrak{D}_q$ defined as

$$\left.\begin{aligned} \mathfrak{D}_p\Phi(x) &\equiv \{\Phi(px)-\Phi(x)\}/x = \Phi_p^{(1)}(x) \quad (p = 1+\epsilon) \\ \mathfrak{D}_q\Phi(x) &\equiv \{\Phi(x)-\Phi(qx)\}/x = \Phi_q^{(1)}(x) \quad (q = 1-\epsilon) \end{aligned}\right\}. \tag{2}$$

In the limit $\epsilon \to 0$ we have

$$\left.\begin{aligned} \lim_{\epsilon\to 0}(p-1)^{-1}\mathfrak{D}_p\Phi(x) &= \lim_{\epsilon\to 0}\{\Phi(x+\epsilon x)-\Phi(x)\}/\epsilon x = \Phi_u'(x) \\ \lim_{\epsilon\to 0}(1-q)^{-1}\mathfrak{D}_q\Phi(x) &= \lim_{\epsilon\to 0}\{\Phi(x)-\Phi(x-\epsilon x)\}/\epsilon x = \Phi_l'(x) \end{aligned}\right\}, \tag{3}$$

giving the differential from upper and lower bounds. When there is no risk of misunderstanding, the subscript letters will be omitted.

In order to obtain the inverse $\mathfrak{D}^{-1}$ in a usable form, I need the symbolic differential operator

$$\theta \equiv x\frac{d}{dx}.$$

Then, since $$f(\theta)x^r = f(r)x^r,$$

we have at once $$q^{n\theta}x^r = q^{nr}x^r,$$

and, if $$\Phi(x) = c_0+c_1x+c_2x^2+\ldots$$

supposed continuous and convergent, then

$$q^\theta\Phi(x) = \Phi(qx), \qquad q^{n\theta}\Phi(x) = \Phi(q^nx).$$

We can write $$\mathfrak{D}_q\Phi(x) = x^{-1}(1-q^\theta)\Phi(x),$$

and obtain the required inverse

$$\begin{aligned} \mathfrak{D}_q^{-1}\Phi(x) &= (1-q^\theta)^{-1}x\Phi(x) \\ &= (1+q^\theta+q^{2\theta}+\ldots)x\Phi(x) \\ &= x\Phi(x)+qx\Phi(qx)+\ldots. \end{aligned}$$

Thus $$(1-q)\mathfrak{D}_q^{-1}\Phi(x) = \{\Phi(x)+q\Phi(qx)+q^2\Phi(q^2x)+\ldots\}(1-q)x, \tag{4}$$

which I write $$\int \Phi(x)\, d(qx). \tag{5}$$

Similarly $$(p-1)\mathfrak{D}_p^{-1}\Phi(x) = \{\Phi(x)+p\Phi(px)+\ldots\}(p-1)x,$$

which is $$\int \Phi(x)\, d(px). \tag{6}$$

These basic integrals are *exactly analogous* to ordinary integrals and from them Riemann's integral is deducible.

3. Riemann's integral and basic integrals

The basic integrals here defined link up with Riemann's definition of an integral. Consider a function $\Phi(x)$ uniform and continuous through an interval $(c, 0)$, where $c > 0$. Write $x_r = cq^r$ $(r = 0,..., \infty)$, where $0 < q < 1$. Dividing the interval at the points x_r we have

$$\int_0^c \Phi(x)\, d(qx) = c(1-q)\{\Phi(c)+q\Phi(cq)+\dots \text{ to } \infty\}$$

$$= \sum_{r=0}^{\infty} (x_r - x_{r+1})\Phi(x_r) \tag{7}$$

and, similarly,

$$\int_0^{cq} \Phi(x)\, d(qx) = c(1-q)\{\Phi(cq)+q\Phi(cq^2)+\dots \text{ to } \infty\}$$

$$= \sum_{r=0}^{\infty} (x_r - x_{r+1})\Phi(x_{r+1}). \tag{8}$$

When $q \to 1$, the lengths $x_r - x_{r+1}$ vanish and the series on the right converge to

$$\int_0^c \Phi(x)\, dx$$

in Riemann's sense.

Similarly, with p-series, we define $x_r = cp^r$ $(r = 0,..., \infty)$ and we have

$$\int_c^{\infty} \Phi(x)\, d(px) = c(p-1)\{\Phi(c)+p\Phi(cp)+\dots \text{ to } \infty\}$$

$$= \sum_{r=0}^{\infty} (x_{r+1} - x_r)\Phi(x_r), \tag{9}$$

$$\int_{cp}^{\infty} \Phi(x)\, d(px) = c(p-1)\{\Phi(cp)+p\Phi(cp^2)+\dots \text{ to } \infty\}$$

$$= \sum_{r=0}^{\infty} (x_{r+1} - x_r)\Phi(x_{r+1}). \tag{10}$$

When $p \to 1$, the lengths of the sub-intervals vanish and the limit on the right is the infinite integral

$$\int_c^{\infty} \Phi(x)\, dx$$

if it exists. Series of type (9) are mentioned in T. J. I'A. Bromwich's *Infinite Series*.*

4. Lemmas

This section contains for reference certain elementary and other known results which are needed in subsequent analysis. Simple algebra is omitted. The reader will find it convenient to pass on to the next

* Second edition 464–5.

section, and refer to the lemmas as required. Dropping the suffix q we have, from (2),

$$\left.\begin{gathered}\mathfrak{D}x^a = (1-q^a)x^{a-1} = [a]x^{a-1}, \qquad \mathfrak{D}^{-1}x^{a-1} = [a]^{-1}x^a, \\ (1+x)_a \equiv \prod_{r=0}^{\infty}(1+xq^r)/(1+xq^{a+r}), \quad \mathfrak{D}(1+x)_a = [a](1+qx)_{a-1}, \\ \mathfrak{D}^{-1}(1+qx)_{a-1} = [a]^{-1}(1+x)_a.\end{gathered}\right\} \tag{11}$$

When a is an integer, the infinite products reduce to finite factorials.

I define $$E(x) \equiv \lim_{n\to\infty}(1+x)(1+qx)\ldots(1+q^{n-1}x)$$

and note that

$$E\{(1-q)x\} = 1+\frac{1-q}{1-q}x+q\frac{(1-q)^2}{(1-q)(1-q^2)}x^2+\ldots, \tag{12}$$

which in the limit, as $q \to 1$, becomes e^x.

$$\mathfrak{D}\{1/E(x)\} = -1/E(x), \qquad \mathfrak{D}^{-1}\{1/E(x)\} = -1/E(x), \tag{13}$$

and reference to (5) gives

$$\int_0^{\infty}\{1/E(x)\}\,d(qx) = [(1-q)\mathfrak{D}^{-1}\{1/E(x)\}]_0^{\infty} = 1-q. \tag{14}$$

I define a basic gamma function as

$$G[a] = \lim_{n\to\infty}\frac{[n-1]!}{[a][a+1]\ldots[a+n-1]}[n]^a, \tag{15}$$

and in terms of this function write Heine's* series and product

$$\sum_{r=0}^{\infty}\frac{[a]_r[b]_r}{[r]![c]_r}q^{r(c-a-b)} = \frac{G[c-a-b]G[c]}{G[c-a]G[c-b]} \quad (a+b-c<0), \tag{16}$$

from which Watson deduces†

$$\sum_{r=0}^{n}\frac{[-n]_r[b]_r}{[r]![c]_r}q^r = q^{nb}\frac{G[c-b+n]G[c]}{G[c-b]G[c+n]} \quad (n \text{ a positive integer}). \tag{17}$$

On replacing $[-n]_r$ by its equivalent $(-)^r[n][n-1]\ldots[n-r+1]q^{-rn-\frac{1}{2}r(r-1)}$, we have

$$\sum_{r=0}^{n}(-)^r\frac{[n]!}{[r]![n-r]!}\frac{[b]_r}{[c]_r}q^{\frac{1}{2}(n-r)(n-r-1)} = \frac{[c-b]_n}{[c]_n}q^{nb+\frac{1}{2}n(n-1)}. \tag{18}$$

Making c infinite and replacing b by a we have

$$\sum_{r=0}^{n}(-)^r\frac{[n]!}{[r]![n-r]!}(1-q^a)_r\,q^{\frac{1}{2}(n-r)(n-r-1)} = q^{na+\frac{1}{2}n(n-1)}. \tag{19}$$

These last two identities are much used in the subsequent analysis.

* E. Heine, *Kugelfunctionen*, 1 (1878), 97.

† G. N. Watson, *Trans. Cambridge Phil. Soc.* 21 (1910), 281.

Integration by parts. Now

$$\begin{aligned}\mathfrak{D}\Phi(x)\Psi(x) &= \{\Phi(x)\Psi(x)-\Phi(qx)\Psi(qx)\}/x \\ &= \{\Phi(x)-\Phi(qx)\}\Psi(x)/x+\{\Psi(x)-\Psi(qx)\}\Phi(qx)/x \\ &\equiv \Phi^{(1)}(x)\Psi(x)+\Psi^{(1)}(x)\Phi(qx), \text{ say.}\end{aligned}$$

I operate with $(1-q)\mathfrak{D}^{-1}$ on both sides of this identity and obtain, by reference to (5),

$$(1-q)\Phi(x)\Psi(x) = \mathsf{S}\,\Phi^{(1)}(x)\Psi(x)\,d(qx)+\mathsf{S}\,\Psi^{(1)}(x)\Phi(qx)\,d(qx). \quad (20)$$

In the limit, as $q \to 1$, this reduces to the formula of ordinary integration by parts.

In (20) replace $\Phi(x)$, $\Phi^{(1)}(x)$, $\Phi(qx)$ by x^a, $[a]x^{a-1}$, $q^a x^a$ respectively, and $\Psi(x)$, $\Psi^{(1)}(x)$ by $1/E(x)$, $-1/E(x)$ respectively as in (13). Then

$$(1-q)\{x^a/E(x)\} = \mathsf{S}\,\{[a]x^{a-1}/E(x)\}\,d(qx)-\mathsf{S}\,\{q^a x^a/E(x)\}\,d(qx). \quad (21)$$

Transformations of basic hypergeometric series. I quote*

$${}_2\Phi_1[\alpha,\beta;\gamma;-x] = \sum_{n=0}^{\infty}\frac{[\alpha]_n[\gamma-\beta]_n}{[n]![\gamma]_n}x^n q^{n\beta+\frac{1}{2}n(n-1)}{}_1\Phi_0[\alpha+n,-x]. \quad (22)$$

This transform is a simple case of a general theorem given by me in previous papers.†

If $\Phi(x) = \sum a_r b_r x^r$ and $f(x) = \sum a_r x^r$,

where both series are absolutely and uniformly convergent, then

$$\Phi(x) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{x^n}{[n]!}D^{(n)}b_0, \quad (23)$$

in which $f^{(n)}(x) = \mathfrak{D}^{(n)}f(x)$,

as defined in (2), and

$$D^{(n)}b_0 = \sum_{r=0}^{n}(-)^r\frac{[n]!}{[r]![n-r]!}b_r q^{\frac{1}{2}(n-r)(n-r-1)}. \quad (24)$$

This theorem, when $q \to 1$, reduces to one given by Euler.

The transform (22) follows at once when we write

$${}_1\Phi_0[a,-x] \text{ for } f(x), \qquad (-)^n[a]_n\,{}_1\Phi_0[x+n,-x] \text{ for } f^{(n)}(x),$$

and note that, with

$$b_0 = 1, \quad b_1 = [\beta]/[\gamma], \quad \ldots, \quad b_n = [\beta]_n/[\gamma]_n,$$

* In more usual notation this would be

$${}_2\Phi_1[q^\alpha, q^\beta; q^\gamma; -x], \text{ etc.}$$

† F. H. Jackson, *Messenger of Math.* **40** (1910), 145; **57** (1928), 169– 7.

we have $$D^{(n)}b_0 = \frac{[\gamma-\beta]_n}{[\gamma]_n} q^{nb+\frac{1}{2}n(n-1)}.$$

It will suffice to state the transform for the series ${}_3\Phi_2$. I use the notation which is customary and convenient in the case of series whose orders are higher than that of ${}_2\Phi_1$.

$${}_3\Phi_2[a_1, a_2, a_3; b_1, b_2; -x]$$

$$= \sum_0^\infty \frac{[a_1]_n[a_2]_n[b_2-a_3]_n}{[n]![b_1]_n[b_2]_n} x^n q^{na_3+\frac{1}{2}n(n-1)} {}_2\Phi_1[a_1+n, a_2+n; b_1+n; -x]. \quad (25)$$

Transformations of this and similar forms, both for single and double variables in ordinary hypergeometric series, have been given by Burchnall and Chaundy in previous numbers of this Journal. The basic forms in double variables have been given by the present writer and W. N. Bailey.

5. Basic infinite integrals

G. H. Hardy* discusses the identity

$$\int_0^\infty \Phi(-x)x^{s-1}\,dx = \Gamma(s)\phi(-s),$$

where $$\Phi(-x) = \phi(0) - \frac{x}{1!}\phi(1) + \frac{x^2}{2!}\phi(2) - \ldots.$$

This extension of Euler's integral for the gamma function was given formally by Ramanujan, but Hardy gives the precise conditions for its validity. He gives also an interesting heuristic deduction, which is valid provided that (i) $\Phi(x)$ is an integral function, (ii) s is positive, (iii) Newton's interpolation theorem

$$\phi(-s) = \phi(0) + \frac{s}{1!}D^1\phi(0) + \frac{s(s+1)}{2!}D^2\phi(0) + \ldots, \quad (26)$$

where

$$D^1\phi(0) = \phi(0) - \phi(1), \quad D^2\phi(0) = \phi(0) - 2\phi(1) + \phi(2), \quad \text{etc.}$$

holds for negative s. In the following sections of this paper I give a basic analogue of Hardy's result, and show its validity in the case of basic hypergeometric functions and others related to them. As a preliminary I need the basic gamma function and integral, which I state as

$$\Gamma[a+1] = (1-q)^{-a-1} \sum_0^\infty \{x^a/E(x)\}\, d(qx), \quad (27)$$

* G. H. Hardy, *Ramanujan* (Cambridge, 1940), 186.

where $$E(x) = 1+\frac{x}{[1]!}+q\frac{x^2}{[2]!}+\dots,$$

as in (12), and always $q < 1$. To obtain this I use (20) for integration by parts. Thus

$$(1-q)[x^a/E(x)]_0^n = [a]\sum_0^n \{x^{a-1}/E(x)\}\,d(qx) - q^a\sum_0^n \{x^a/E(x)\}\,d(qx). \tag{28}$$

When $n \to \infty$, the expression on the left vanishes at both limits, giving

$$\sum_0^\infty \{x^a/E(x)\}\,d(qx) = q^{-a}[a]\sum_0^\infty \{x^{a-1}/E(x)\}\,d(qx). \tag{29}$$

These infinite integrals are convergent, as is easily seen when we refer to (14) and write the integral on the left as an infinite series

$$(1-q)\lim_{n\to\infty}\sum_{r=0}^{\infty} q^{r(a+1)}n^{a+1}/E(q^rn) \quad (a > 0). \tag{30}$$

Here, when r and n each tend to infinity,

$$q^rn \to 0, \qquad q^{r(a+1)}n^{a+1} \to 0, \qquad E(q^rn) \to 1,$$

and, if u_r denote the $(r+1)$th term of the series,

$$u_r \to 0, \qquad u_{r+1}/u_r \to q^a < 1,$$

which proves convergence.

I now define a function $\mathfrak{G}[a+1]$ by a convergent integral: thus

$$\left.\begin{aligned}\mathfrak{G}[a+1] &= \sum_0^\infty \{x^a/E(x)\}\,d(qx) = q^{-a}[a]\mathfrak{G}[a]\\ \mathfrak{G}[a+n] &= q^{-na+\frac{1}{2}n(n-1)}[a][a-1]\dots[a-n+1]\mathfrak{G}[a]\end{aligned}\right\}. \tag{31}$$

The series (30) shows that the $\mathfrak{G}$ functions exist only when $q < 1$. In order to get the *proper analogue* of the ordinary gamma function I write

$$\Gamma[a+1] = (1-q)^{-a-1}\mathfrak{G}[a+1], \tag{32}$$

and show that this function $\Gamma[a+1]$ exists when $q \leqslant 1$, and then, as $q \to 1$, reduces to Euler's gamma function.

(i) I consider firstly the case when a is a positive integer. Repetition of (31) gives

$$\mathfrak{G}[a+1] = q^{-\frac{1}{2}a(a+1)}[a][a-1]\dots[1]\sum_0^\infty \{1/E(x)\}\,d(qx). \tag{33}$$

By (14), the integral on the right is $1-q$, so that we can write

$$\Gamma[a+1] = (1-q)^{-a-1}\mathfrak{G}[a+1] = q^{-\frac{1}{2}a(a+1)}\frac{[a]!}{(1-q)^a}. \tag{34}$$

In the limit, as $q \to 1$, where a is an integer,

$$\Gamma[a+1] = a! = \Gamma(a+1). \tag{35}$$

(ii) When a is any positive number, consider

$$(1-q)^{-a-1}\sum_0^n \{x^a/E(x)\}\,d(qx). \tag{36}$$

I change the variable to τ, where

$$x = (1-q)\tau = \epsilon\tau = d(q\tau).$$

The upper limit of the integral must now be changed to n/ϵ, and

$$d(qx) = (1-q)^2\tau = (1-q)\,d(q\tau).$$

Simple algebra gives us

$$\sum_0^{n/\epsilon} [\tau^a/E\{(1-q)\tau\}]\,d(q\tau). \tag{37}$$

In the limit, as $q \to 1$, $E\{(1-q)\tau\} \to e^\tau$, by (12), and by § 3 the basic integral becomes an ordinary integral. Then, as $n \to \infty$, we have

$$\int_0^{\infty^2} \tau^a e^{-\tau}\,d\tau = \Gamma(a+1). \tag{38}$$

The ∞^2 of the upper limit is interesting.

Factorial expression for $\Gamma[a+1]$.

Assuming, as is reasonable, that the basic function has an equivalent expression analogous to Euler's well-known factorial, then such basic factorial, when a is an integer, must reduce to (34) with the solitary factor $q^{-\frac{1}{2}a(a+1)}$; also in the limit, as $q \to 1$, it must reduce to Euler's factorial. I write therefore

$$\Gamma[a+1] = \lim_{n\to\infty} \frac{[n]![n]^a q^{-\frac{1}{2}a(a+1)}}{[a+1]\ldots[a+n](1-q)^a} \quad (q \leqslant 1). \tag{39}$$

Simple algebra gives

$$\Gamma[a+1]/\Gamma[a] = \left(\lim_{n\to\infty} \frac{1-q^n}{1-q^{a+n}}\right)\frac{[a]}{1-q}q^{-a}, \tag{40}$$

which, since $[a] = 1-q^a$, reduces in the limit to $q^{-a}(1-q^a)/(1-q)$, while

$$\mathfrak{G}[a+1]/\mathfrak{G}[a] = (1-q^a)q^{-a} \quad (q < 1).$$

In the final section of the paper I give examples (72), (73) of a direct reduction of a basic integral to a factorial form, but so far I have not succeeded in reducing them to $\Gamma[a+1]$. I now leave the basic gamma function and proceed to the more general integrals.

6. Integrals of more general type

$$\sum_0^\infty \Phi(-x)x^{s-1}\,d(qx) = \mathfrak{G}[s]\phi(-s) \quad (q < 1), \tag{41}$$

$$(1-q)^{-a-1}\sum_0^\infty \Phi(-x)x^{s-1}\,d(qx) = \Gamma[s]\phi(-s) \quad (q \leqslant 1), \tag{42}$$

where $$\Phi(-x) = \phi(0) - \frac{x}{[1]!}\phi(1) + \frac{x^2}{[2]!}\phi(2) - \ldots,$$

supposed absolutely and uniformly convergent. I form the product of two absolutely convergent series

$$\Phi(-x)E(x) = \left\{\sum_{r=0}^{\infty}(-)^r\frac{x^n}{[n]!}\phi(r)\right\}\left\{\sum_{r=0}^{\infty}\frac{x^n}{[n]!}q^{\frac{1}{2}n(n-1)}\right\} = \sum_{n=0}^{\infty}\frac{x^n}{[n]!}D^{(n)}\phi(0), \tag{43}$$

in which the differences $D^{(1)}$, $D^{(2)}$,... are as defined in (24), i.e.

$$D^{(1)}\phi(0) = \phi(0) - \phi(1), \qquad D^{(2)}\phi(0) = q\phi(0) - [2,1]\phi(1) + \phi(2),$$

$$D^{(n)}\phi(0) = \sum_{r=0}^{n}(-)^r[n,r]\phi(r)q^{\frac{1}{2}(n-r)(n-r-1)}. \tag{44}$$

The series (43) can be absolutely convergent in the case of certain forms of coefficients $\phi(0)$, $\phi(1)$,.... In order to show that this is possible, I consider a simple basic hypergeometric series

$${}_1\Phi_0[a,-x] = 1 - \frac{[a]}{[1]!}x + \frac{[a][a+1]}{[2]!}x^2 - \ldots. \tag{45}$$

Here $$\phi(0) = 1, \quad \phi(1) = 1 - q^a, \quad \ldots, \quad \phi(n) = [a]_n,$$

$$D^{(1)}\phi(0) = q^a, \quad D^{(2)}\phi(0) = q^{2a+1}, \quad \ldots,$$

$$D^{(n)}\phi(0) = \sum(-)^r[n,r]\phi(r)q^{\frac{1}{2}(n-r)(n-r-1)} = q^{na+\frac{1}{2}n(n-1)},$$

by (19). Then

$${}_1\Phi_0[a,-x] \times E(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]!}q^{na+\frac{1}{2}n(n-1)}, \tag{46}$$

which is absolutely convergent when $q < 1$ and a is positive.

With this example to justify the assumption that (43) may converge for all real x, I transform (43) by transferring $E(x)$ from the left to the right of (43), and, after multiplying both sides by x^{s-1}, write the basic integrals

$$\int_0^{\infty}\Phi(-x)x^{s-1}\,d(qx) = \sum_{n=0}^{\infty}\frac{D^{(n)}\phi(0)}{[n]!}\int_0^{\infty}\frac{x^{n+s-1}}{E(x)}\,d(qx). \tag{47}$$

The change in the order of integration and summation is justifiable since the series is supposed absolutely and uniformly convergent, and the terms are finite. Reference to the basic function (31) gives

$$\int_0^{\infty}\Phi(-x)x^{s-1}\,d(qx) = \sum_{n=0}^{\infty}\mathfrak{G}[n+s]\frac{D^{(n)}\phi(0)}{[n]!}, \tag{48}$$

which is

$$\mathfrak{G}[s]\left\{\phi(0)+q^{-s}\frac{[s]}{[1]!}D^{(1)}\phi(0)+\ldots+q^{-ns+\frac{1}{2}n(n-1)}\frac{[s]_n}{[n]!}D^{(n)}\phi(0)+\ldots\right\}. \quad (49)$$

This formal result is the basic analogue of that given by Hardy (loc. cit.). Inspection of the series of differences shows that, if $D^{(n)}\phi(0)$ reduces to some form such as $f(n)q^{\lambda n+\kappa n(n-1)}$ ($\lambda > s$, $\kappa \geqslant \frac{1}{2}$), then the series will be convergent, since $[s]_n$, $[n]!$ are finite for all n when $q < 1$, provided that $f(n)$ is finite for all n. In the examples which follow there is convergence of this kind and the limit of the series of differences is shown to be $\phi(-s)$.

7. An example

$$\sum_0^\infty {}_1\Phi_0[a, -x]x^{s-1}\,d(qx) = \mathfrak{G}[s]\phi(-s).$$

Reference to (44) enables us to write

$$\sum_0^\infty {}_1\Phi_0[a, -x]x^{s-1}\,d(qx) = \sum_{n=0}^\infty \frac{q^{na+\frac{1}{2}n(n-1)}}{[n]!}\sum_0^\infty \frac{x^{n+s-1}}{E(x)}\,d(qx), \quad (50)$$

which, as (31) shows, is

$$\sum_{n=0}^\infty \frac{q^{na+\frac{1}{2}n(n-1)}}{[n]!}\mathfrak{G}[n+s].$$

Now $$\mathfrak{G}[n+s] = q^{-ns+\frac{1}{2}n(n-1)}[s]_n\mathfrak{G}[s].$$

We then have

$$\mathfrak{G}[s]\left\{1+q^{a-s}\frac{[s]}{[1]!}+q^{2(a-s)}\frac{[s][s+1]}{[2]!}+\ldots\right\} = \mathfrak{G}[s]G[a-s]/G[a].^* \quad (51)$$

Now reference to the series (45) gives $[a]_n = G[a+n]/G[a]$, so that

$$\mathfrak{G}[s]G[a-s]/G[a] = \mathfrak{G}[s]\phi(-s).$$

In this special case the basic analogue of Newton's interpolation theorem holds for negative s. In the final section of the paper I give a formal proof of the interpolation theorem in general.

8. A corollary

In (51), if we replace a by $a+r$, s by $s+r$, we have

$$\sum_0^\infty {}_1\Phi_0[a+r, -x]x^{s+r-1}\,d(qx) = \mathfrak{G}[s+r]G[a-s]/G[a+r]. \quad (52)$$

Now $$\mathfrak{G}[s+r] = q^{-rs-\frac{1}{2}r(r-1)}[s]_r\,\mathfrak{G}[s],$$

and $$G[a+r] = [a]_r\,G[a].$$

* E. Heine quoted by W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge, 1935), 66 (4).

Thus

$$\int_0^\infty {}_1\Phi_0[a+r, -x]x^{s+r-1}\,d(qx) = q^{-rs-\frac{1}{2}r(r-1)}\frac{[s]_r\,G[a-s]}{[a]_r\,G[a]}\,\mathfrak{G}[s]$$

$$= \frac{[s]_r}{[a]_r}q^{-rs-\frac{1}{2}r(r-1)}\int_0^\infty {}_1\Phi_0[a, -x]x^{s-1}\,d(qx). \quad (53)$$

This corollary is important and fundamental in the next example, where I consider the basic integral for ${}_2\Phi_1$ and deduce that

$$\int_0^\infty {}_2\Phi_1[a, b; c; -x]x^{s-1}\,d(qx) = \mathfrak{G}[s]\phi(-s), \quad (54)$$

where $$\phi(n) = [a]_n[b]_n/[c]_n.$$

Reference to (53) gives

$$\int_0^\infty {}_1\Phi_0[a+r, -x]x^{r+s-1}\,d(qx) = \mathfrak{G}[s]\frac{[s]_r\,G[a-s]}{[a]_r\,G[a]}q^{-rs-\frac{1}{2}r(r-1)}. \quad (55)$$

I now multiply both sides by an arbitrary factor λ_r, and sum in r, which gives us formally

$$\sum_0^\infty \lambda_r \int_0^\infty {}_1\Phi_0[a+r, -x]x^{r+s-1}\,d(qx) = \sum_{r=0}^\infty \lambda_r\frac{[s]_r}{[a]_r}q^{-rs-\frac{1}{2}r(r-1)}\mathfrak{G}[s]\frac{G[a-s]}{G[a]}. \quad (56)$$

Now, since $[s]_r$, $[a]_r$ are finite for all r, we can choose λ_r to make the series on the right converge with any degree of rapidity we wish. Guided by inspection of the transformation (22) I choose

$$\lambda_r = \frac{[a]_r[c-b]_r}{[r]![c]_r}q^{rb+\frac{1}{2}r(r-1)}, \quad (57)$$

which gives us

$$\sum_{r=0}^\infty \frac{[a]_r[c-b]_r}{[r]![c]_r}q^{rb+\frac{1}{2}r(r-1)}\int_0^\infty {}_1\Phi_0[a+r, -x]x^{r+s-1}\,d(qx)$$

$$= \mathfrak{G}[s]\frac{G[a-s]}{G[a]}\sum_{r=0}^\infty \frac{[s]_r[c-b]_r}{[r]![c]_r}q^{r(b-s)} \quad (b-s>0). \quad (58)$$

The series on the left is the basic integral of the transform (22) for ${}_2\Phi_1[a, b; c; -x]x^{s-1}$. The series in the expression on the right is a summable Heine's series (16). We then have

$$\int_0^\infty {}_2\Phi_1[a, b; c; -x]x^{s-1}\,d(qx)$$

$$= \mathfrak{G}[s]\frac{G[a-s]G[b-s]G[c]}{G[a]G[b]G[c-s]} = \mathfrak{G}[s]\phi(-s), \quad (59)$$

since $$\phi(n) = \frac{[a]_n[b]_n}{[c]_n} = \frac{G[a+n]G[b+n]G[c]}{G[a]G[b]G[c+n]},$$

which justifies the heuristic general form (48), in the case of the basic hypergeometric series ${}_2\Phi_1$. By means of a corollary we can proceed to the case of ${}_3\Phi_2$, which follows.

9. The function ${}_3\Phi_2$

I consider
$$\int_0^\infty {}_3\Phi_2[a_1, a_2, a_3; b_1, b_2; -x]x^{s-1}\, d(qx), \tag{60}$$

where the change of notation is convenient and usual in the case of hypergeometric series of order higher than that of ${}_2\Phi_1$.

In (59) I replace a by a_1+r, b by a_2+r, c by b_1+r, s by $s+r$, and, referring to (33), I replace

$$\mathfrak{G}[s+r] \quad \text{by} \quad q^{-rs-\frac{1}{2}r(r-1)}[s]_r\, \mathfrak{G}[s],$$
$$G[a_1+r] \quad \text{by} \quad [a_1]_r G[a_1],$$

and so on, giving

$$\int_0^\infty {}_2\Phi_1[a_1+r, a_2+r; b_1+r; -x]x^{r+s-1}\, d(qx)$$
$$= \mathfrak{G}[s]\frac{[s]_r[b_1]_r}{[a_1]_r[a_2]_r}\, q^{-rs-\frac{1}{2}r(r-1)}\frac{G_1[a_1-s]G[a_2-s]G[b_1]}{G[a_1]G[a_2]G[b_1-s]}. \tag{61}$$

Here again, as in (56), I multiply both sides by an arbitrary λ_r and form infinite series summable in r. Inspection of the transforms (22), (25) determines our choice of λ_r so as to give a convergent summable Heine's series, namely

$$\lambda_r = \frac{[a_1]_r[a_2]_r[b_2-a_3]_r}{[r]![b_1]_r[b_2]_r}\, q^{ra_3+\frac{1}{2}r(r-1)},$$

which gives us

$$\sum_{r=0}^\infty \frac{[a_1]_r[a_2]_r[b_2-a_3]_r}{[r]![b_1]_r[b_2]_r}\, q^{ra_3+\frac{1}{2}r(r-1)} \int_0^\infty {}_2\Phi_1[a_1+r, a_2+r; b_1+r; -x]x^{r+s-1}\, d(qx)$$
$$= \mathfrak{G}[s]\frac{G[a_1-s]G[a_2-s]G[b_1]}{G[a_1]G[a_2]G[b_1-s]} \sum_{r=0}^\infty q^{r(a_3-s)}\frac{[s]_r[b_2-a_3]_r}{[r]![b_2]_r}. \tag{62}$$

The series on the right is Heine's series and summable by (16). We have

$$\mathfrak{G}[s]\frac{G[a_1-s]G[a_2-s]G[a_3-s]G[b_1]G[b_2]}{G[a_1]G[a_2]G[a_3]G[b_1-s]G[b_2-s]} = \mathfrak{G}[s]\phi(-s). \tag{63}$$

It is obvious that by proceeding in this way step by step we can next obtain

$$\int_0^\infty {}_4\Phi_3[a_1, a_2, a_3, a_4; b_1, b_2, b_3; -x]x^{s-1}\, d(qx) = \mathfrak{G}[s]\phi(-s),$$

and so on in succession for basic hypergeometric series of any order.

10. In this section I obtain integrals of another form connected with the basic analogues of Saalschütz and other identities. Consider

$$\int_0^\infty {}_2\Phi_1[a,b;c;-x]\left\{\lambda_0-\lambda_1\frac{x}{[1]!}+\ldots(-)^N\lambda_N\frac{x^N}{[N]!}\right\}x^{s-1}\,d(qx). \qquad (64)$$

In (59) on replacing s by $r+s$ we have

$$\int_0^\infty {}_2\Phi_1[a,b;c;-x]x^{r+s-1}\,d(qx) = \mathfrak{G}[s+r]\frac{G[a-s-r]G[b-s-r]G[c]}{G[a]G[b]G[c-s-r]}$$

so long as $a > s+r$, $b > s+r$.

Term-by-term integration of the finite series of basic integrals in (64) gives us

$$\sum_{r=0}^{N}(-)^r\mathfrak{G}[s+r]\frac{G[a-s-r]G[b-s-r]G[c]\lambda_r}{G[a]G[b]G[c-s-r][r]!}. \qquad (65)$$

By various choices of λ_r we can convert this series so as to be summable by basic Heine or Saalschütz or other finite identities. Now

$$\begin{aligned}G[a-s] &= [a-s-1][a-s-2]\ldots[a-s-r]G[a-s-r]\\ &= (-)^r q^{r(a-s)-\frac{1}{2}r(r+1)}[1+s-a]_r\,G[a-s-r],\end{aligned}$$

with similar expressions for parameters b, c. Also

$$\mathfrak{G}[s+r] = q^{-rs-\frac{1}{2}r(r-1)}[s]_r\mathfrak{G}[s].$$

I now express (64) as

$$\mathfrak{G}[s]\frac{G[a-s]G[b-s]G[c]}{G[a]G[b]G[c-s]}\sum_{r=0}^{N}\frac{[s]_r[1+s-c]_r\lambda_r}{[r]![1+s-a]_r[1+s-b]_r}q^{r(c-a-b+1)}. \qquad (66)$$

Inspection of the finite series shows that we can convert it into a basic Saalschützian by taking

$$\lambda_r = (-N)_r q^{r(c-a-b)}, \qquad N = a+b-c.$$

This reduces (66)* to

$$\mathfrak{G}[s]\frac{G[a-s]G[b-s]G[c]}{G[a]G[b]G[c-s]}\frac{[c-a]_N[1-a]_N}{[c-a-s]_N[1+s-a]_N}. \qquad (67)$$

Replacing N by integer $a+b-c$ and cancelling common factors we have

$$\mathfrak{G}[s]\frac{G[a-s]G[c]G[b-c+1]G[1+s-a]G[c-a-s]}{G[a]G[c-s]G[b-c+s+1]G[1-a]G[c-a]}. \qquad (68)$$

In the same way by other choice of λ_r and series of forms ${}_3\Phi_2$, ${}_4\Phi_3$,... we can obtain various relations with basic Dougall series and products. The formulae are very lengthy but may be worth study.

* W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge, 1935), 68.

11. In this section I give several simple examples of basic integrals which are of some interest in relation to the basic gamma function and theta functions. I indicate proofs very briefly. Reference to (10), (11) and integration by parts gives us three formulae

$$\{x^a(1-x)_n\}_0^1 = \int_0^1 [a]x^{a-1}(1-qx)_n\,d(qx) - \int_0^1 [n](1-qx)_{n-1}x^a\,d(qx), \quad (69)$$

$$\{x^n/(1+x)_{a+n}\}_0^\infty = \int_0^\infty \{[n]x^{n-1}/(1+qx)_{a+n}\}\,d(qx) - \int_0^\infty \{x^n[a+n]/(1+qx)_{a+n+1}\}\,d(qx), \quad (70)$$

$$\{x^r\log x\}_0^1 = \int_0^1 \{[r]x^{r-1}\log x\}\,d(qx) - q^r\int_0^1 \{x^{r-1}\log q\}\,d(qx). \quad (71)$$

In each case the function on the left vanishes at both limits. From (69) by giving n successively values $n-1$, $n-2$,..., 1, also in succession replacing a by $a+1$, $a+2$,..., $a+n-1$ and then cancelling common factors, I obtain

$$\int_0^1 x^{a-1}(1-qx)_n\,d(qx) = [n]!/\{[a][a+1]...[a+n]\}. \quad (72)$$

Similarly, from (70),

$$\int_0^\infty \{x^n/(1+qx)_{a+n+1}\}\,d(qx) = [n]!/\{[a][a+1]...[a+n]\}. \quad (73)$$

From (71), $$\int_0^1 \{x^{r-1}\log x\}\,d(qx) = q^r\log q/(1-q^r)^2, \quad (74)$$

whence

$$\int_0^1 \{\log x/(1-qx)\}\,d(qx) = \log q\left\{\frac{q}{(1-q)^2}+\frac{q^3}{(1-q^2)^2}+\frac{q^5}{(1-q^3)^2}+...\right\}. \quad (75)$$

12. In *Fundamenta Nova* Jacobi gives several identities of which the following is a typical example

$$\frac{q}{1-q^2}+\frac{8q^2}{1-q^4}+\frac{27q^3}{1-q^6}+... = \frac{q(1+4q+q^2)}{(1-q)^4}+\frac{q^3(1+4q^3+q^6)}{(1-q^3)^4}+.... \quad (76)$$

They are all special cases of the following simple theorem of basic integration. Let $\Phi(x) = c_0+c_1x+c_2x^2+...$ $(x<1)$ and

$$\Phi(x)+q\Phi(qx)+q^2\Phi(q^2x)+... \quad (0<q<1) \quad (77)$$

be absolutely and uniformly convergent. Basic integration of $\Phi(x)$ and reference to (4) give us

$$\int \Phi(x)\,d(qx) = (1-q)x\{\Phi(x)+q\Phi(qx)+...\} = (1-q)x\{c_0/(1-q)+c_1x/(1-q^2)+...\}. \quad (78)$$

After removing the common factors $(1-q)x$ from both sides, and again integrating we have

$$(1-q)x\{\Phi(x)+2q\Phi(qx)+3q^2\Phi(q^2x)+\ldots\} = (1-q)x\{c_0/(1-q)^2+c_1x/(1-q^2)^2+\ldots\} \quad (79)$$

and similarly after removing the common factor, and again integrating, we get

$$(1-q)x\{\Phi(x)+3q\Phi(qx)+6q^2\Phi(q^2x)+\ldots\} = (1-q)x\{c_0/(1-q)^3+c_1x/(1-q^2)^3+\ldots\} \quad (80)$$

and, in general, after m integrations

$$\Phi(x)+\frac{(m)_1}{1!}q\Phi(qx)+\frac{(m)_2}{2!}q^2\Phi(q^2x)+\ldots = c_0/(1-q)^m+c_1x/(1-q^2)^m+\ldots. \quad (81)$$

This *trivial* theorem gives rise to many interesting identities when we apply it to any of the well-known elementary expansions

$$\left.\begin{aligned} 1/(1-x)^2 &= 1+2x+3x^2\ldots,\\ (1+x)/(1-x)^3 &= 1^2+2^2x+\ldots,\\ (1+4x+x^2)/(1-x)^4 &= 1^3+2^3x+\ldots,\\ (1+11x+11x^2+x^3)/(1-x)^5 &= 1^4+2^4x+\ldots,\\ (1+x^2)/(1-x^2)^2 &= 1+3x^2+5x^4+\ldots,\\ (1+6x^2+x^4)/(1-x^2)^3 &= 1+3^2x^2+5^2x^4+\ldots,\\ (1+23x^2+23x^4+x^6)/(1-x^2)^4 &= 1+3^3x^2+\ldots. \end{aligned}\right\} \quad (82)$$

The general forms are well known and need not be given here. The reader can easily obtain

$$\frac{1-6q^2+q^4}{(1+q^2)^3}q-\frac{1-6q^4+q^8}{(1+q^4)^3}q^2+\ldots = \frac{q}{1+q}-\frac{9q^3}{1+q^3}+\frac{25q^5}{1+q^5}-\ldots, \quad (83)$$

$$\frac{1+x^2}{(1-x^2)^2}+\frac{1+q^2x^2}{(1-q^2x^2)^2}q+\frac{1+q^4x^2}{(1-q^4x^2)^2}q^2-\ldots = \frac{1}{1-q}+\frac{3q^2x^3}{1-q^3}+\frac{5q^4x^5}{1-q^5}+\ldots, \quad (84)$$

$$\sum_0^q \frac{1}{1+x^2}\,d(qx) = \frac{q}{1-q}+\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}+\ldots = \tfrac{1}{4}\sum_{n=1}^{\infty} r(n)q^n, \quad (85)$$

which enumerates the representations of n as sum of two squares.* It is possible that the more complex identities obtainable in this way might give interesting results in theory of partitions of numbers. This is outside the scope of the present paper.

* Hardy and Wright, *Theory of Numbers* (Oxford, 1938), 257.

13. The interpolation theorem

Referring to (49) let us suppose that $\phi(-s)$ can be expressed as

$$c_0+c_1 q^{-s}\frac{[s]}{[1]!}+c_2 q^{1-2s}\frac{[s][s+1]}{[2]!}+\ldots$$

where c_0, c_1,... are independent of s. If now we give to s the values $0, -1, -2,\ldots$ in succession, we obtain equations

$$\phi(0)=c_0, \qquad \phi(1)=c_0-c_1, \qquad \phi(2)=c_0-[2,1]c_1+c_2, \qquad \ldots,$$
$$\phi(n)=c_0-[n,1]c_1+[n,2]c_2-\ldots(-)^n c_n.$$

If now we multiply both sides of the equation for $\phi(r)$ by

$$(-)^r[n,r]q^{\frac{1}{2}(n-r)(n-r-1)} \quad (r=0,\,1,\,2,\ldots,\,n)$$

and sum by columns, all columns on the right vanish by Heine's sum except the single term c_n, giving us

$$\sum_0^n (-)^r[n,r]\phi(r)q^{\frac{1}{2}(n-r)(n-r-1)}=c_n=D^{(n)}\phi(0).$$

This *a priori* result is strengthened, however, since we have shown that in certain cases the infinite series of differences does in fact converge to $\phi(-s)$.

To add interest to this present paper I state here a theorem given by me nearly forty years ago* in the Quarterly Journal.

$$\mathsf{S}\,\Phi(x)\,d(px)=(1+\tfrac{1}{2}\epsilon-\tfrac{1}{12}\epsilon^2+\ldots)\int\Phi(x)\,dx-$$
$$-\tfrac{1}{2}\epsilon x\Phi(x)+\sum(-)^{r+1}\frac{B_r}{(2r)!}(-\log^{2r-1}p)\theta^{2r-1}x\phi(x),$$

where B_r is the rth Bernoulli number and $p=1+\epsilon$, with a similar theorem for base $q<1$ ($q=1-\epsilon$), by change of sign of ϵ. If now

$$\Phi(x)=c_0+c_1x+c_2x^2+\ldots,$$

where c_r, x^r are positive real numbers for all r, $\Phi(x)$ is finite or convergent, and $\epsilon<1$, then these expansions can be applied to give numerical approximations connecting the basic (p, q) integrals with the ordinary integral. These formal theorems may have an interest in connexion with (9) and the Riemann integral, as showing in the case of a very limited class of functions how the integral is approached from upper and lower bounds.

[At Dr. Jackson's express request I have edited this with some freedom; in particular § 3 has been rewritten. Dr. Jackson wishes to express his thanks and I to explain my responsibility.—T. W. Chaundy.]

* F. H. Jackson, *Quart. J. of Math.* (1910), 195.

ON THE CRITICAL LATTICES OF THE 'QUADRIFOIL'

By H. AP SIMON (*Oxford*)

[Received 15 September 1949]

1. Introduction

It has been proved† that, if the region $\mathscr{T}$ is a star domain (**1**) enclosed in a finite closed star domain $\mathscr{H}$, then‡ $\Delta(\mathscr{H}) \geqslant \Delta(\mathscr{T})$. If for all $\mathscr{T}$ contained in but different from $\mathscr{H}$ the strict inequality $\Delta(\mathscr{H}) > \Delta(\mathscr{T})$ holds, then $\mathscr{H}$ is said to be *irreducible*.

In this note I define first a 'quadrifoil' region $\mathscr{R}_a$ bounded by the arcs of four circles, and construct from it a region $\mathscr{H}_a$. I show that $\mathscr{H}_a$ is an irreducible region, and that, when $0 < a \leqslant \sqrt{\frac{1}{5}}$, it is enclosed in $\mathscr{R}_a$; I also find the critical lattices of $\mathscr{R}_a$ for these values of a. I then show that the critical lattices of $\mathscr{R}_a (\sqrt{\frac{1}{5}} < a \leqslant 1)$ can be deduced from those of the region $\mathscr{R}_{1/\sqrt{5}}$.

2. The region $\mathscr{R}_a$

The region $\mathscr{R}_a$ is defined as the sum of the regions bounded by the four circles

$$\left.\begin{aligned} (\mathscr{S}_1) \quad & (x-a)^2+y^2 = 1 \\ (\mathscr{S}_2) \quad & (y-a)^2+x^2 = 1 \end{aligned}\right\}$$

and $\mathscr{S}'_1, \mathscr{S}'_2$, their respective reflections in the origin, where $0 < a \leqslant 1$. (See Figs. 2, 3.) (Throughout, dashes denote reflection in the origin.)

Call A that point of intersection of $\mathscr{S}_1, \mathscr{S}_2$ and B that point of intersection of $\mathscr{S}_1, \mathscr{S}'_2$ which is on the boundary of $\mathscr{R}_a$. Take L on $\mathscr{S}_2$, M on $\mathscr{S}_1$ such that L, M are on the boundary of $\mathscr{R}_a$, and LM is equal and parallel to OB; take N on $\mathscr{S}_1$, T on $\mathscr{S}'_2$, such that N, T are on the boundary of $\mathscr{R}_a$, and NT is equal and parallel to AO. (By symmetry, N and T are the reflections of M and L respectively in the x-axis.)

Now A is the point

$$[\tfrac{1}{2}\{a+\sqrt{(2-a^2)}\},\ \tfrac{1}{2}\{a+\sqrt{(2-a^2)}\}],$$

† K. Mahler (**1**), Theorem 2.

‡ I use Mahler's notation, now in general use. A lattice Λ of determinant $d(\Lambda)$ is said to be an *admissible* lattice of a bounded star domain $\mathscr{R}$ if it has no point other than the origin in the interior of $\mathscr{R}$. The (attained) minimum of $d(\Lambda)$ extended over all the admissible lattices of $\mathscr{R}$ is denoted by $\Delta(\mathscr{R})$, and all admissible lattices of determinant $\Delta(\mathscr{R})$ are called *critical*.

Quart. J. Math. Oxford (2), 2 (1951), 17–25

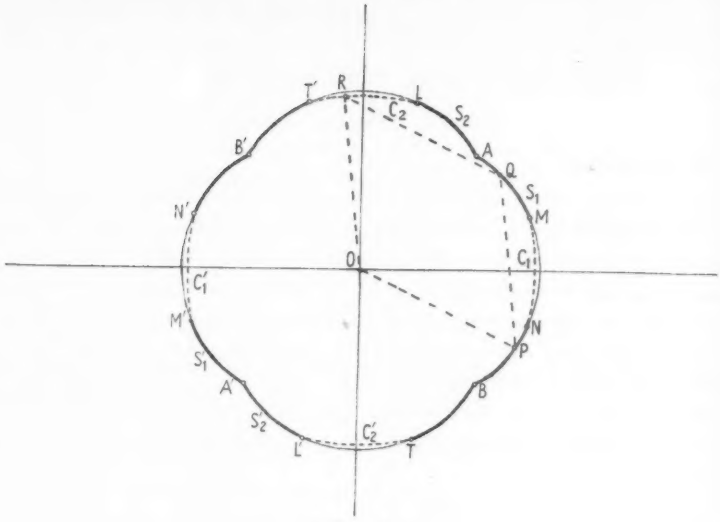
T'
R
L
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FIG. 1.

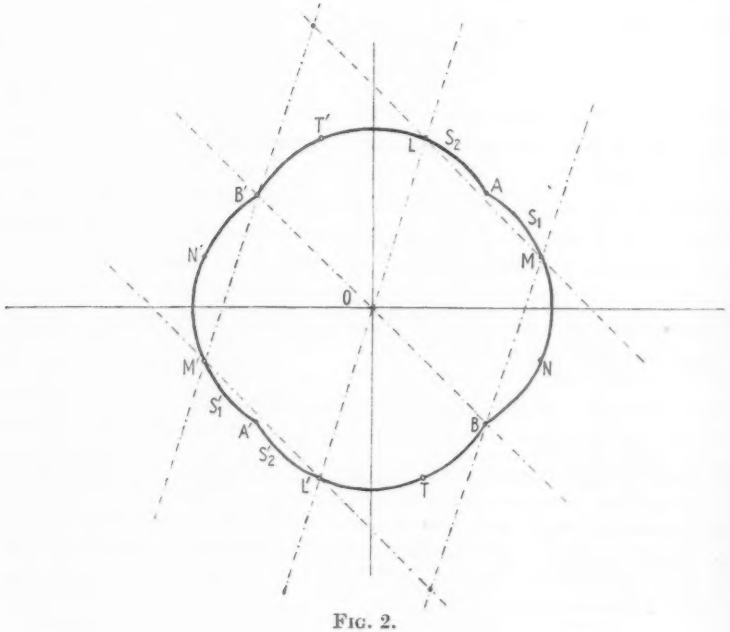
T'
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FIG. 2.

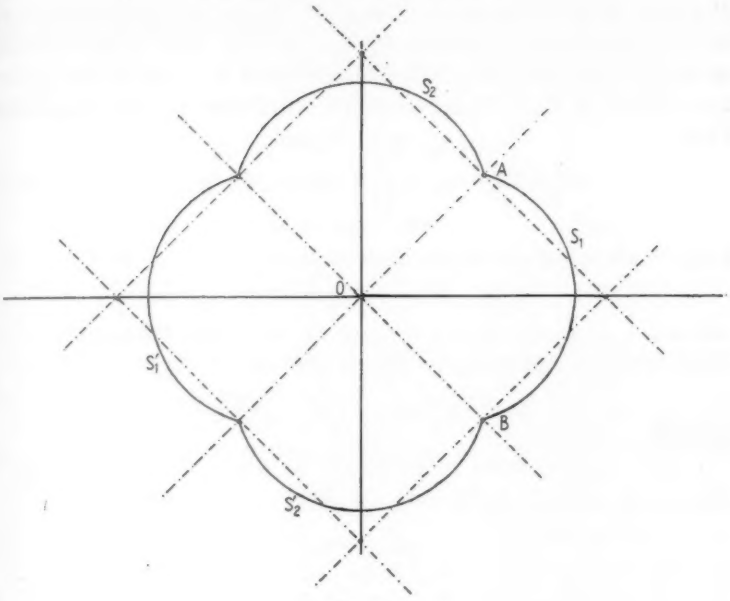

FIG. 3.

and hence we find that L is the point (X, Y), where

$$X = \tfrac{1}{4}[a-\sqrt{(2-a^2)}+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}],$$
$$Y = \tfrac{1}{4}[3a+\sqrt{(2-a^2)}+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}].$$

3. Construction of $\mathscr{H}_a$

The region $\mathscr{H}_a$ is defined only for $0 < a \leqslant \sqrt{\frac{1}{5}}$. I take as its boundary those parts of the boundary of $\mathscr{R}_a$ between MA, AL, $T'B', B'N'$, $M'A', A'L'$, TB, BN, together with curves $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}'_1$, $\mathscr{C}'_2$, between NM, LT', $N'M'$, $L'T$ respectively, to be defined.

Let $\mathscr{A}$ be the area of the parallelogram $OBML$. Then we find that

$$\mathscr{A} = \tfrac{1}{4}\{a+\sqrt{(2-a^2)}\}[2a+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}].$$

It may be verified that, for $a \leqslant \sqrt{\frac{1}{5}}$, all triangles OPA for P in the arc BN have area greater than or equal to $\frac{1}{2}\mathscr{A}$, and all triangles OPM have area less than or equal to $\frac{1}{2}\mathscr{A}$. Hence for any P in the arc BN there is a point Q in the arc MA such that the area of the triangle OPQ is $\frac{1}{2}\mathscr{A}$ (see Fig. 1).

Since LM is equal and parallel to OB, when P is at B, Q is at M and $(Q-P)$ is at L; and, since AT' is equal and parallel to NO, when

P is at N, Q is at A and $(Q-P)$ is at T'. It has been shown that for P in BN (part of the boundary $\mathscr{H}_a$) Q is in MA (part of the boundary of $\mathscr{H}_a$). I define the curve $\mathscr{C}_2$ to be the locus of $R = (Q-P)$ as P moves from B to N. If P, Q, R have respective coordinates (x_1, y_1), (x_2, y_2), (x, y), then

$$x = x_2 - x_1, \qquad y = y_2 - y_1,$$
$$(x_1-a)^2+y_1^2 = 1, \qquad (x_2-a)^2+y_2^2 = 1,$$
$$x_1 y_2 - x_2 y_1 = \mathscr{A}.$$

From these equations we can show that

$$\tfrac{1}{4}(x^2+y^2)^2 = (x^2+y^2)-(\mathscr{A}-ay)^2,$$

which is a bicircular quartic through T', L. I take the arc $T'L$ of this curve to be $\mathscr{C}_2$, and similarly the arc MN of

$$\tfrac{1}{4}(x^2+y^2)^2 = (x^2+y^2)-(\mathscr{A}-ax)^2$$

to be $\mathscr{C}_1$.

I now mention three properties of $\mathscr{C}_2$ which may be obtained by elementary algebra: (i) the line $y = x\tan\alpha$ cuts $\mathscr{C}_2$ just once if

$$|\tan\alpha| \geqslant \frac{X}{Y},$$

and not at all if

$$|\tan\alpha| < \frac{X}{Y};$$

since $\mathscr{R}_a$ is a star domain, it follows that $\mathscr{H}_a$ is a star domain; (ii) $\mathscr{C}_2$ lies above the straight line $T'L$; (iii) $\mathscr{C}_2$ lies below the arc $T'L$ of the circle centre O and radius OL. (I use (ii) in § 4 and (iii) in §§ 4, 5, 7.)

4. In this section I show that all points of the parallelogram $OPQR$ other than P, Q, R are inner points of $\mathscr{H}_a$, except in the particular case in which $a = \sqrt{\frac{1}{5}}$ and P is at B, for which the point A is on QR $(= LM)$.

$\mathscr{H}_a$ is a star domain, and so all points of the line segments OP, OR are inner points of $\mathscr{H}_a$; the line segment PQ is a chord of $\mathscr{S}_1$ all points of which have x-coordinate less than that of MN, so that (by § 3 (ii)) all inner points of the line segment PQ are inner points of $\mathscr{H}_a$. It remains to consider inner points of the line segment QR.

The point A is further than QR from OP if

$$\begin{vmatrix} \frac{1}{2}\{a+\sqrt{(2-a^2)}\} & \frac{1}{2}\{a+\sqrt{(2-a^2)}\} & 1 \\ x & y & 1 \\ x_2 & y_2 & 1 \end{vmatrix} > 0,$$

i.e. if

$$\tfrac{1}{2}\{a+\sqrt{(2-a^2)}\}(x_1-y_1) > \mathscr{A},$$

i.e. if

$$x_1 - y_1 > X+Y.$$

Now, if P is an inner point of BN,

$$-\tfrac{1}{2}\{a+\sqrt{(2-a^2)}\} < y_1 < -X,$$

and so

$$\begin{aligned} x_1-y_1 &= \sqrt{(1-y_1^2)}-y_1+a \\ &> \min\{X+Y, a+\sqrt{(2-a^2)}\}; \end{aligned}$$

hence, as $X+Y \leqslant a+\sqrt{(2-a^2)}$ for $a \leqslant \sqrt{\frac{1}{5}}$, it follows that

$$x_1-y_1 > X+Y.$$

Also, if P is at B, $y_1 = -\frac{1}{2}\{a+\sqrt{(2-a^2)}\}$, and so

$$x_1-y_1 \geqslant X+Y,$$

with equality only when $a = \sqrt{\frac{1}{5}}$.

Hence A is outside the parallelogram $OPQR$ except in the particular case in which $a = \sqrt{\frac{1}{5}}$ and P is at B, for which A is on QR.

It follows that as R is an inner point of $\mathscr{S}_2$ (§ 3 (iii)) QR does not intersect the bounding arcs QA, AL (except at A in the particular case mentioned).

It also follows that QR intersects $\mathscr{S}_1$ at Q and at an inner point of QR, and so no point of QR outside the line segment QR lies on $\mathscr{S}_1$. Suppose QR intersected the open arc LR in an inner point of QR, say R'. Then by the definition of $\mathscr{C}_2$ the point $Q' = R'+P$ would be on $\mathscr{S}_1$, and, as RQ is parallel to OP, Q' would be a point of QR outside the line segment QR. This contradicts the hypothesis, and so QR does not intersect the bounding arc RL other than at R.

Hence all inner points of QR are inner points of $\mathscr{H}_a$, except A in the particular case mentioned; and so the proof is complete that all points of the parallelogram $OPQR$ other than P, Q, R are inner points of $\mathscr{H}_a$, except in the particular case in which $a = \sqrt{\frac{1}{5}}$ and P is at B, for which the point A is on QR.

5. In this section I show that $\Delta(\mathscr{H}_a) = \mathscr{A}$, and that every point on the boundary of $\mathscr{H}_a$ is a point of a critical lattice of $\mathscr{H}_a$.

5.1. The region $\mathscr{H}_a$ is a closed and finite star domain (§ 4 (i)), so that† any critical lattice of $\mathscr{H}_a$ has at least two lattice points on the boundary of $\mathscr{H}_a$. By symmetry, we need only consider those lattices having a point, P say, in the arc BN or an inner point of the arc $T'L$.

5.2. *The lattice Λ having base points P, R is admissible*

The points $\pm P$, $\pm R$, $\pm(P+R)$ lie on the boundary of $\mathscr{H}_a$. Since the y-coordinates of $-P$, R are greater than or equal to X, Y respec-

† K. Mahler (**1**), Theorem 6.

tively, and, as $X+Y > 1+a$, the points $\pm(R-P)$ lie outside the circle

$$x^2+y^2 = (1+a)^2,$$

and so, as $\mathscr{H}_a$ is contained in this circle, outside $\mathscr{H}_a$.

It may be verified that for $a \leqslant \sqrt{\frac{1}{5}}$ the lines $T'M$, AB do not intersect the circle $x^2+y^2 = \frac{1}{4}(1+a)^2$, and so any line parallel to OP through $\pm nR$, and any line parallel to OR through $\pm nP$, where $n \geqslant 2$, do not intersect the circle $x^2+y^2 = (1+a)^2$. As $\mathscr{H}_a$ lies within this circle the only lattice points of Λ which can be on or inside $\mathscr{H}_a$ are $\pm P$, $\pm R$, $\pm(P \pm R)$, which have already been shown to be on or outside $\mathscr{H}_a$. Hence Λ is an admissible lattice of $\mathscr{H}_a$.

5.3. In this section I show that the lattice Λ having base points P, R is critical.

If $a < \sqrt{\frac{1}{5}}$, all points of the parallelogram $OPQR$ other than P, Q, R are inner points of $\mathscr{H}_a$ (§ 4), and (§ 3) the lattice Λ having base points P, R has determinant $d(\Lambda) = \mathscr{A}$. Hence† any lattice having a point at P (or R) and determinant less than $\mathscr{A}$ is inadmissible. Since (§ 5.2) Λ is admissible for all P in BN (and R in $T'L$) it follows that the lattices are critical, and $\Delta(\mathscr{H}_a) = \mathscr{A}$.

If $a = \sqrt{\frac{1}{5}}$, the same applies, except that when P is at B the point A is on QR, and so the lattice Λ' having base points A, B is critical if it is admissible. This is clearly the case, as all points of Λ' other than $\pm A$, $\pm B$ lie on or outside the circle $x^2+y^2 = \{a+\sqrt{(2-a^2)}\}$, which contains $\mathscr{H}_a$.

Since P takes all positions in BN, and R all positions in $T'L$, it follows that $\Delta(\mathscr{H}_a) = \mathscr{A}$ and all points of the boundary of $\mathscr{H}_a$ are points of regular‡ critical lattices of $\mathscr{H}_a$, and further that there are no other critical lattices except in the case $a = \sqrt{\frac{1}{5}}$, when there is a singular critical lattice having base points A, B.

6. In this section I prove that $\mathscr{H}_a$ is irreducible.

As has been shown (§ 4), the line segment QR cuts $\mathscr{S}_1$ in Q and an inner point of QR. Hence there is a point Q'' on the boundary of $\mathscr{H}_a$ such that $|Q-Q''| < \epsilon$ where ϵ is an arbitrary small positive number, and Q'' is nearer than Q to OP. Call Λ'' the lattice having base points P, Q''. Then $d(\Lambda'') < d(\Lambda)$, and, as (§ 5.2) the only points of Λ on the boundary of $\mathscr{H}_a$ are $\pm P$, $\pm Q$, $\pm R$, for sufficiently small ϵ the only points of Λ'' inside $\mathscr{H}_a$ are $\pm R''$, where $R'' = Q''-P$; and, since

$$|Q-Q''| < \epsilon, \qquad |R-R''| < \epsilon.$$

† K. Ollerenshaw (**3**), Lemma.

‡ K. Mahler (**1**), 135.

Therefore R'' is an irreducible point.† Similarly it may be shown that Q is an irreducible point. Hence, as Q takes all positions in AL and R all positions in $T'L$, all points on the boundary of $\mathscr{H}_a$ are irreducible points. It follows‡ that $\mathscr{H}_a$ is an irreducible region. We have, then,

THEOREM 1. *The region $\mathscr{H}_a$ is irreducible, and its critical lattices have determinant*

$$\Delta(\mathscr{H}_a) = \tfrac{1}{4}\{a+\sqrt{(2-a^2)}\}[2a+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}];$$

when $0 < a < \sqrt{\frac{1}{5}}$ the only critical lattices are regular; when $a = \sqrt{\frac{1}{5}}$ there is also the singular critical lattice having base points at A and B.

7. The Region $\mathscr{R}_a$ for $0 < a \leqslant \sqrt{\frac{1}{5}}$

It has been shown in § 3 (iii) that $\mathscr{C}_2$ lies within $\mathscr{S}_2$; so that $\mathscr{H}_a$ is contained in $\mathscr{R}_a$. Then§ any critical lattice of $\mathscr{H}_a$ which is an admissible lattice of $\mathscr{R}_a$ is a critical lattice of $\mathscr{R}_a$. The lattice Λ_1 having base points B, M is an admissible lattice of $\mathscr{R}_a$; for by § 5.2 all its points other than $\pm B, \pm M, \pm L$ are outside the circle $x^2+y^2 = (1+a)^2$, which circle contains $\mathscr{R}_a$. By § 5.3 any critical lattice of $\mathscr{H}_a$, other than Λ_1 and its reflection in the x-axis, has a point which is an inner point of $\mathscr{C}_2$ (and so of $\mathscr{R}_a$), except in the case of $a = \sqrt{\frac{1}{5}}$, when there is a singular critical lattice having base points A, B. Theorem 2 follows.

THEOREM 2. *The region $\mathscr{R}_a$ $(0 < a \leqslant \sqrt{\frac{1}{5}})$ has just two critical lattices, of determinant*

$$\tfrac{1}{4}\{a+\sqrt{(2-a^2)}\}[2a+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}],$$

one defined by

$$x = \tfrac{1}{4}[2a+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}](\xi+\eta)+\tfrac{1}{4}\{a+\sqrt{(2-a^2)}\}(\xi-\eta),$$

$$y = \tfrac{1}{4}[2a+\sqrt{\{6+2a\sqrt{(2-a^2)}\}}](\xi+\eta)-\tfrac{1}{4}\{a+\sqrt{(2-a^2)}\}(\xi-\eta),$$

and the other its reflection in the x-axis. In the particular case $a = \sqrt{\frac{1}{5}}$ there is also the third critical lattice defined by

$$x = 2\sqrt{\tfrac{1}{5}}(\xi+\eta), \qquad y = 2\sqrt{\tfrac{1}{5}}(\xi-\eta).$$

8. I now consider the critical lattices of $\mathscr{R}_a$ when $\sqrt{\frac{1}{5}} < a < 1$. If we replace $\mathscr{S}_1$ of $\mathscr{R}_{1/\sqrt{5}}$ by $\mathscr{S}_1^*$, where $\mathscr{S}_1^*$ is a circle with its centre at $(b, 0)$ and passing through A and B; and also replace $\mathscr{S}_2$ by $\mathscr{S}_2^*$, where $\mathscr{S}_2^*$ is a circle with centre $(0, b)$ and passing through A and B'; and similarly with $\mathscr{S}'_1$ and $\mathscr{S}'_2$, then we have a region similar (homothetic) to $\mathscr{R}_a$ (for some value of a, not equal to $\sqrt{\frac{1}{5}}$ if $b \neq a$), and completely containing

† Rogers (**2**), Theorem 3.

‡ Rogers (**2**), Theorem 1.

§ K. Mahler (**1**), Theorem 2.

$\mathscr{R}_a$ if $\sqrt{\frac{1}{5}} < b < \sqrt{\frac{4}{5}}$. If, then, we take $\sqrt{\frac{1}{5}} < b < 2\sqrt{\frac{1}{5}}$, and define $\mathscr{R}_a^*$ as the sum of the regions bounded by the circles $\mathscr{S}_1^*$, $\mathscr{S}_2^*$, $\mathscr{S}_1'^*$, $\mathscr{S}_2'^*$, a critical lattice of $\mathscr{R}_{1/\sqrt{5}}$ is also a critical lattice of $\mathscr{R}_a^*$ if it remains admissible for this larger region $\mathscr{R}_a^*$.

The circle $\mathscr{S}_1^*$ has as its equation

$$(x-b)^2+y^2 = b^2-4\sqrt{\tfrac{1}{5}}\,b+\tfrac{8}{5},$$

and the lattice $\mathscr{L}$ defined by

$$x = 2\sqrt{\tfrac{1}{5}}(\xi+\eta), \qquad y = 2\sqrt{\tfrac{1}{5}}(\xi-\eta),$$

(a critical lattice of $\mathscr{R}_{1/\sqrt{5}}$) is still admissible if b is such that the point $(4\sqrt{\frac{1}{5}}, 0)$ is external to $\mathscr{S}_1^*$; i.e. if

$$(4\sqrt{\tfrac{1}{5}}-b)^2 \geqslant b^2-4\sqrt{\tfrac{1}{5}}\,b+\tfrac{8}{5},$$

i.e. if $$b \leqslant 2\sqrt{\tfrac{1}{5}}.$$

If $b > 2\sqrt{\frac{1}{5}}$, the point $(4\sqrt{\frac{1}{5}}, 0)$ is inside the region; and if $b \leqslant 2\sqrt{\frac{1}{5}}$, the lattice is admissible if

$$\{2\sqrt{\tfrac{1}{5}}(\xi+\eta)-b\}^2+\{2\sqrt{\tfrac{1}{5}}(\xi-\eta)\}^2 \geqslant b^2-4\sqrt{\tfrac{1}{5}}\,b+\tfrac{8}{5}$$

for all integer ξ, η, not both zero, i.e. if

$$\tfrac{8}{5}(\xi^2+\eta^2-1) \geqslant 4\sqrt{\tfrac{1}{5}}(\xi+\eta-1)b,$$

which is the case for $b \leqslant 2\sqrt{\frac{1}{5}}$, since $\xi^2+\eta^2-1 \geqslant \xi+\eta-1$ for integer values of ξ, η.

I now express $\mathscr{R}_a^*$ as a region which is the sum of the regions enclosed by the circles $(x\pm a)^2+y^2 = 1, \qquad (y\pm a)^2+x^2 = 1;$

to do so we expand the region $\mathscr{R}_a^*$ and its critical lattice $\mathscr{L}$ by the homothetic stretch

$$(x, y) \to (\lambda x, \lambda y)$$

where $$\lambda^2 = b^2-4\sqrt{\tfrac{1}{5}}\,b+\tfrac{8}{5},$$

so that the region is now the sum of the regions enclosed by the circles

$$\left(x\pm\frac{b}{\lambda}\right)^2+y^2 = 1, \qquad \left(y\pm\frac{b}{\lambda}\right)^2+x^2 = 1,$$

and its critical lattice is defined by

$$x = 2\frac{1}{\lambda}\sqrt{\tfrac{1}{5}}(\xi+\eta), \qquad y = 2\frac{1}{\lambda}\sqrt{\tfrac{1}{5}}(\xi-\eta).$$

Take $$b = \frac{2a}{(1-a^2)\sqrt{5}}\{-a+\sqrt{(2-a)}\};$$

so that we find

$$b = \lambda a, \qquad \frac{1}{\lambda} = \tfrac{1}{4}\sqrt{5}\{a+\sqrt{(2-a^2)}\},$$

and the condition $\sqrt{\frac{1}{5}} < b < 2\sqrt{\frac{1}{5}}$ gives $\sqrt{\frac{1}{5}} < a < 1$.

Hence, if $\sqrt{\frac{1}{5}} < a < 1$, the critical lattice of the region which is the sum of the regions

$$(x \pm a)^2 + y^2 \leqslant 1, \qquad (y \pm a)^2 + x^2 \leqslant 1$$

is defined by

$$x = \tfrac{1}{2}\{a+\sqrt{(2-a^2)}\}(\xi+\eta),$$
$$y = \tfrac{1}{2}\{a+\sqrt{(2-a^2)}\}(\xi-\eta).$$

This completes the proof of Theorem 3.

THEOREM 3. *The region* $\mathscr{R}_a$ $(\sqrt{\frac{1}{5}} < a \leqslant 1)$ *has one critical lattice of determinant*

$$1+a\sqrt{(2-a^2)},$$

defined by

$$x = \tfrac{1}{2}\{a+\sqrt{(2-a^2)}\}(\xi+\eta),$$
$$y = \tfrac{1}{2}\{a+\sqrt{(2-a^2)}\}(\xi-\eta).$$

(*This lattice is singular when* $a < 1$; *regular when* $a = 1$.)

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ON A COEFFICIENT PROBLEM OF LITTLEWOOD AND SOME TRIGONOMETRICAL SUMS

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1. We denote by $Q(z)$ any function which is regular and never takes any value $\pm 2n\pi i$ $(n = 0, 1, 2,...)$ in the circle $|z| < 1$. Clearly the necessary and sufficient condition for a function to be a Q is that it should be of the form $\log P$, where P is a 'Picard' function regular and never taking the values 0 or 1 in the unit circle. It is well known† that $Q(z)$ is subordinate in $|z| < 1$ to the function

$$\bar{Q}(z) = \bar{Q}(z, q_0)$$

uniquely determined, for given $q_0 = Q(0)$, by the relations‡

$$\bar{Q}(z) = \log \lambda(\tau), \qquad z = -\frac{\tau - \tau_0}{\tau - \bar{\tau}_0}, \tag{1.1}$$

$$\log \lambda(\tau_0) = q_0, \tag{1.2}$$

where

$$-1 < \mathscr{R}\tau_0 \leqslant 1, \quad \mathscr{I}\tau_0 > 0, \quad |\tau_0 - \tfrac{1}{2}| \geqslant \tfrac{1}{2}, \quad |\tau_0 + \tfrac{1}{2}| > \tfrac{1}{2}, \tag{1.3}$$

and the determination of the logarithm is fixed by (1.2). Here $\lambda(\tau) = k^2(\tau)$ is the elliptic modular function. We denote the coefficients of z^N in $Q(z)$ and $\bar{Q}(z)$ by q_N and $\bar{q}_N$ respectively.§

In the present paper, I propose to study the order of magnitude of q_N and $\bar{q}_N$ as $N \to \infty$.

Consider first the sub-class of the functions Q consisting of those functions

$$B(z) = \sum b_N z^N$$

regular in $|z| < 1$ and taking there no value w for which $\mathscr{R}w = 0$. It is evident that B is the class of functions which are regular and have either a positive or a negative real part in the unit circle, and it is familiar that for such functions the following relations hold:‖

$$|B(z)| \leqslant A(b_0)(1-\rho)^{-1} \quad (|z| = \rho < 1),$$

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |B(\rho e^{i\theta})|^{\lambda}\, d\theta\right)^{1/\lambda} \leqslant A(b_0, \lambda)(1-\rho)^{-1+1/\lambda} \quad (\rho < 1,\ \lambda > 1),$$

$$|b_N| \leqslant A(b_0).$$

† See e.g. Littlewood (**9**), Theorem 231. I have written $-z$ in place of Littlewood's z. $\bar{Q}$ is, of course, a Q. ‡ $\bar{\tau}_0$ is the conjugate of τ_0. § $\bar{q}_0 = q_0$.

‖ I use $A(b, c,...)$ to denote a positive constant depending only on b, c,... not necessarily the same on any two occurrences: in particular, A will denote a positive absolute constant.

The first and second of these results hold for the wider class of functions Q,† and it is natural to inquire whether this parallelism holds also for the third: that is whether

$$|q_N| \leqslant A(q_0). \tag{1.4}$$

This result, however, is false for general q_N, and it seems unlikely to be true for $\bar{q}_N$, though this I am unable to prove.

To prove that (1.4) does not hold in general, consider the function

$$\chi(z) = -16 \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m+1)\{1+z^{2(2m+1)}\}}. \tag{1.5}$$

It is known‡ that $\chi(z)$ never takes any value $\pm 2n\pi i$ for $z \neq 0$. It differs from a Q by taking one of the forbidden values (viz. 0) just once. The function

$$\chi_1(z) = \tfrac{1}{2}\chi(z) + \pi i$$

is therefore a Q, and it is easy to see that for N of the form $\prod p$, the product being taken over all primes p of the form $4k+1$ not exceeding x, the Nth coefficient of χ_1 is greater than $A \operatorname{logloglog} N$. For, expanding the expression (1.5) we have

$$\chi_1(z) = -8 \sum_{m,n=0}^{\infty} \frac{(-1)^n}{2m+1} z^{(2m+1)(2n+1)} + \pi i. \tag{1.6}$$

If N has the above form, a term in z^N occurs in (1.6) whenever $2m+1$ is a prime p of the form $4k+1$ not exceeding x or a product of such primes, and it is easily seen that all these terms have the same sign. The coefficient of z^N in χ_1 is therefore greater than

$$8 \sum_{\substack{p \leqslant x \\ p \equiv 1 (\mathrm{mod}\ 4)}} \frac{1}{p} \geqslant A \operatorname{loglog} x.$$

Since

$$\log N = \sum_{\substack{p \leqslant x \\ p \equiv 1 (\mathrm{mod}\ 4)}} \log p \leqslant Ax,$$

the result now follows.

In the related problem of finding an upper bound for general q_N, I have nothing new to add to what is already known:§ namely that

$$|q_N| < A(q_0) \log N \quad (N > 1). \tag{1.7}$$

† Lévy (**6**), Littlewood (**7**), Stein (**11**).

‡ Littlewood (**9**), 204.

§ Ibid. 200. The result (1.7) is deduced by a subordination argument from the inequality

$$\int_{-\pi}^{\pi} |\mathscr{R}\bar{Q}(\rho e^{i\theta})|\, d\theta \leqslant A(q_0) \log\Big(\frac{1}{1-\rho}\Big)$$

[and this latter result cannot be improved. See Nevanlinna (**10**), § 72. An alternative proof runs on the lines of a discussion of a similar problem by Littlewood (**8**)].

In the special case of $\bar{q}_N$, however, it is possible to improve the result (1.7) to

$$|\bar{q}_N| < A(q_0)(\log N \operatorname{loglog} N)^{\frac{1}{2}}, \tag{1.8}$$

and most of the remainder of this paper is devoted to the proof of this result.

The arrangement of the paper is as follows. For simplicity I consider the special case $\tau_0 = i$.† In §§ 2–5 we obtain an expression for $\bar{q}_N$ as an infinite series, namely‡

$$\bar{q}_N = 2i(-1)^N + \frac{32}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1} \cos[2N \arctan\{(2m-1)/n\}]}{n^2+(2m-1)^2}. \tag{1.9}$$

The proof of this formula given in the original version of this paper made use of the theory of the boundary values of a positive harmonic function. A much simpler proof was suggested to the writer by Mr. A. E. Ingham who has kindly allowed it to be reproduced here. Since the original proof retains some points of interest, however, I give also a shortened version of this. In §§ 6–13 I prove the inequality (1.8). The proof depends on the application to the series (1.9) of two theorems on the estimation of trigonometrical sums. The first of these theorems is an immediate corollary of a theorem of van der Corput, while the second is a modification of a recent theorem of Vinogradow.§ The proof of this latter theorem is given in Appendix I.

In § 14 we consider the sum

$$\bar{S}_N = \bar{q}_1 + \bar{q}_2 + \ldots + \bar{q}_N,$$

and prove that

$$\bar{S}_N = O(\log^2 N) \tag{1.10}$$

as $N \to \infty$. This result is easily deduced from the formula (1.9), and it does not seem possible to improve it by the deep methods required in the proof of (1.8).

The paper concludes (Appendix II) with a brief note on Hecke's zeta function $\zeta(s; \lambda^k)$ in the algebraic field $k(i)$.

This problem was suggested to me by Professor J. E. Littlewood, and I wish to record my thanks for the help and advice he has given me during the preparation of this paper.

† See (1.1).

‡ The series on the right of (1.9) is not absolutely convergent, and is to be interpreted, here and elsewhere, as $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \ldots \right)$.

§ See § 9.

PROOF OF THE FORMULA (1.9)

2. We start from the formula

$$\kappa(\tau) = \log\lambda(\tau) = 4\log 2 + \pi\tau i + 8\sum_{n=1}^{\infty}\frac{(-1)^n e^{n\pi\tau i}}{n(1+e^{n\pi\tau i})}, \tag{2.1}$$

valid for a certain determination of the logarithm. Since $\bar{Q}(0) = q_0$, we have, from the definition (1.1) of $\bar{Q}$,

$$\bar{Q}(z) = q_0 - \kappa(\tau_0) + \kappa(\tau), \tag{2.2}$$

where

$$\tau = \frac{\tau_0 + \bar{\tau}_0 z}{1+z}$$

and τ_0 satisfies (1.3).

Let $q = e^{\pi\tau i}$, and write

$$f_n(z) = \frac{1-q^n}{1+q^n} = \sum_{N=0}^{\infty} a_N(n) z^N \quad (|z| < 1). \tag{2.3}$$

Since

$$\frac{q^n}{1+q^n} = \tfrac{1}{2}\{1 - f_n(z)\} = \tfrac{1}{2}\{1 - a_0(n)\} - \tfrac{1}{2}\sum_{N=1}^{\infty} a_N(n) z^N,$$

and the series on the right of (2.1) is uniformly convergent in

$$|z| \leqslant \rho < 1,$$

we have, from (2.2),

$$\bar{q}_N = (-1)^N(\tau_0 - \bar{\tau}_0) + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} a_N(n) \quad (N \geqslant 1). \tag{2.4}$$

3. We now determine the coefficient $a_N = a_N(n)$ for a fixed value of n. For simplicity, consider only the special case $\tau_0 = i$, when

$$f(z) = f_n(z) = \frac{1-q^n}{1+q^n}, \qquad q = \exp\left\{-\pi\left(\frac{1-z}{1+z}\right)\right\}.$$

Ingham's proof is now as follows. Let

$$Z = n\left(\frac{1-z}{1+z}\right), \qquad \phi_m = \arctan\frac{2m-1}{n}.$$

Then we have

$$\begin{aligned} f(z) = \tanh\tfrac{1}{2}\pi Z &= \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{Z}{Z^2+(2m-1)^2} \\ &= \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{n(1-z^2)}{n^2(1-z)^2+(2m-1)^2(1+z)^2} \\ &= \frac{4n}{\pi}\sum_{m=1}^{\infty}\frac{1-z^2}{(1-2z\cos 2\phi_m+z^2)\{n^2+(2m-1)^2\}}, \end{aligned}$$

and on expanding the Poisson kernel (the double series obtained is

absolutely convergent for $|z| < 1$) we obtain

$$a_N = \frac{8n}{\pi} \sum_{m=1}^{\infty} \frac{\cos(2N\phi_m)}{n^2+(2m-1)^2}. \tag{3.1}$$

The formula (1.9) now follows from (3.1) and (2.4).

In the general case we may make the obvious modifications in the foregoing analysis, and we then obtain, with $\tau_0 = s_0 + it_0$ subject to (1.3),

$$\tilde{q}_N = 2it_0(-1)^N + \frac{16t_0}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n-1} \cos[2N \arctan\{(2m-1-ns_0)/nt_0\}]}{(nt_0)^2+(2m-1-ns_0)^2}.$$

4. The original proof is much deeper and appeals to the general theory of positive harmonic functions. We again consider the special case $\tau_0 = i$. As before, let $\phi_m = \arctan\{(2m-1)/n\}$.

The following facts are easily verified:

(*a*) $f(z) = f_n(z)$ is regular in and on $|z| = 1$ save for simple poles at the points $e^{2i\phi_m}$ ($m = 0, \pm 1, \ldots$) and an essential singularity at $z = -1$;

(*b*) as z tends radially to the point $e^{2i\phi_m}$

$$\mathscr{R}f(z) = \frac{4n}{\pi\{n^2+(2m-1)^2\}(1-|z|)} + O(1);\dagger$$

(*c*) as z tends radially to any point $e^{i\theta}$ for which $2\phi_m < \theta < 2\phi_{m+1}$ $\mathscr{R}f(z) \to 0$, and uniformly in any closed interval contained in this interval.‡

Now $$u(\rho, \theta) = \mathscr{R}f(z) = a_0 + \sum_{N=1}^{\infty} a_N \rho^N \cos N\theta.$$

Let $$u_1(\rho, \theta) = \sum_{N=1}^{\infty} \frac{1}{n} a_N \rho^N \sin N\theta,$$

so that§ $$u_1(\rho, \theta) = \int_0^{\theta} \{u(\rho, \phi) - a_0\}\, d\phi = \int_0^{\theta} u(\rho, \phi)\, d\phi - a_0\theta.$$

† The constant of the O is independent of $|z|$. It may depend on m, n, and τ_0, but this is immaterial here.

‡ At such points $e^{i\theta}$, $f(z)$ is regular, $\mathscr{R}f(z) = 0$, and the uniformity follows from the uniform continuity of $f(z)$.

§ Here a_N is real, and $u(\rho, \theta)$ is an even function of θ. In the general case let

$$u(\rho, \theta) = \mathscr{R}f(z) = b_0 + \sum_{1}^{\infty} (b_N \cos N\theta + c_N \sin N\theta)\rho^N,$$

say, and then let

$$u_1(\rho, \theta) = \sum_{1}^{\infty} \frac{1}{N} (-c_N \cos N\theta + b_N \sin N\theta)\rho^N.$$

Then since $\int_{-\pi}^{\pi} u_1(\rho, \theta)\, d\theta = 0$, u_1 must vanish for some $\theta_0 = \theta_0(\rho)$, and

$$u_1(\rho, \theta) = \int_{\theta_0}^{\theta} \{u(\rho, \phi) - b_0\}\, d\phi.$$

Since $u(\rho, \theta) \geqslant 0$, it is known† that

(i) $u_1(\rho, \theta)$ tends radially everywhere to a function $U_1(\theta)$ of bounded variation;

(ii)‡
$$u(\rho, \theta) - a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \psi - \theta)\, dU_1(\psi);$$

(iii)
$$a_N = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos N\psi\, dU_1(\psi) \quad (N \geqslant 1).$$

If now θ_1, θ_2 are any two points such that $2\phi_m < \theta_2 < \theta_1 < 2\phi_{m+1}$,

$$u_1(\rho, \theta_1) - u_1(\rho, \theta_2) + a_0(\theta_1 - \theta_2) = \int_{\theta_2}^{\theta_1} u(\rho, \phi)\, d\phi \to 0$$

as $\rho \to 1$, in virtue of (c), and therefore

$$U_1(\theta_1) + a_0\theta_1 = U_1(\theta_2) + a_0\theta_2.$$

Hence $U_1(\psi) + a_0\psi$ *is a step-function with jumps, of magnitude* I_m *say, at each of the points* $\psi = 2\phi_m$ $(m = 0, \pm 1, \pm 2, \ldots)$.

Since $\int P\, d\psi = 2\pi$, we have from (ii)

$$u(\rho, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \psi - \theta)\, d\{U_1(\psi) + a_0\psi\},$$

and, when $\theta = 2\phi_m$,

$$u(\rho, \theta) = \frac{1}{2\pi}\left(\frac{2I_m}{1-\rho}\right) + O(1)$$

as $\rho \to 1$. Comparing this with (b) we find that

$$I_m = \frac{4n}{n^2 + (2m-1)^2}.$$

Finally, from (iii),

$$a_N = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos N\psi\, dU_1(\psi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos N\psi\, d\{U_1(\psi) + a_0\psi\}$$

$$= \frac{4n}{\pi} \sum_{m=-\infty}^{\infty} \frac{\cos 2N\phi_m}{n^2 + (2m-1)^2} = \frac{8n}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2N\phi_m}{n^2 + (2m-1)^2}, \qquad (4.1)$$

from which (1.9) follows as before.

† See e.g. Zygmund (**18**), § 4.36.

‡ $P(\rho, \theta)$ is Poisson's kernel $(1-\rho^2)/(1 - 2\rho\cos\theta + \rho^2)$

5. There is an alternative method of obtaining U_1 which seems of interest. We have

$$u_1 = \int_0^\theta u\, d\phi - a_0\theta.$$

If C denotes the arc of the circle $z = \rho e^{i\phi}$ from $\phi = 0$ to $\phi = \theta$ (in the sense of increasing ϕ),

$$\int_C \frac{f(z)}{z}\, dz = i\int_0^\theta (u+iv)\, d\phi,$$

so that

$$u_1(\rho,\theta) = \mathscr{I}\int_C \frac{f(z)}{z}\, dz - a_0\theta.$$

If θ is not one of the values $2\phi_m$ ($m = 0, \pm 1, \ldots$), the arc C can be deformed into the contour formed by the segments $\rho \leqslant r \leqslant 1$ of the radii at $\phi = 0$ and $\phi = \theta$ together with the arc of the unit circle from $\phi = 0$ to $\phi = \theta$ indented at each of the points $\phi = 2\phi_m$ on this arc. Each of these points is a pole of $f(z)$ and we therefore have

$$\begin{aligned} U_1(\theta) &= \lim_{\rho\to 1-} u_1(\rho,\theta) \\ &= -\pi \sum \{\text{real parts of the residues of } f(z)/z \text{ at poles}\} - a_0\theta, \end{aligned}$$

where the sum is taken over all poles $e^{2i\phi_m}$ for which $0 < 2\phi_m < \theta$. $U_1 + a_0\theta$ is clearly a step-function with a jump of magnitude

$$-\pi\{\text{real part of the residue of } f(z)/z \text{ at } e^{2i\phi_m}\}$$

at each of the points $\theta = 2\phi_m$. The residue is easily found to be

$$-4n/\pi\{n^2+(2m-1)^2\},$$

and (4.1) now follows as before.

6. Before passing on to the proof of (1.8), we observe that methods similar to those in §§ 3–5 may be applied to the function

$$\log k'^2(\tau) = \log\{1-k^2(\tau)\} = -16\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(2n+1)(1+q^{2(2n+1)})} \qquad (q = e^{\pi i\tau})$$

(this relation being valid for a certain determination of the logarithm). As before, with $\tau = (\tau_0+\bar{\tau}_0 z)/(1+z)$, this function is a Q, and in the (simplest) special case in which $\tau_0 = \frac{1}{2}i$ we have, for its Nth coefficient q_N,

$$q_N = -128\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m \sin[2N \arctan\{(2m+1)/(2n+1)\}]}{(2m+1)^2+(2n+1)^2}.$$

PROOF OF THE INEQUALITY $|\bar{q}_N| \leqslant A(q_0)(\log N \log\log N)^{\frac{1}{2}}$

7. For simplicity we consider only the special case $\tau_0 = i$ when, from (2.4),

$$\bar{q}_N = 2i(-1)^N + \frac{32}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}\cos(2\pi w_{n,m})}{n^2+(2m-1)^2}, \tag{7.1}$$

where
$$w_{n,m} = \frac{N}{\pi}\arctan\frac{2m-1}{n}.$$

We write also
$$u_{n,m} = \frac{(-1)^{n-1}\cos(2\pi w_{n,m})}{n^2+(2m-1)^2},$$

$$v_{n,m} = \frac{(-1)^{n-1}\exp(2\pi i w_{n,m})}{n^2+(2m-1)^2}.$$

In the proof of the inequality above we actually prove four results which may be summed up in the following lemmas.†

LEMMA 1. $$\sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty} u_{n,m} = O(1)$$

as $N \to \infty$.

LEMMA 2. $$\sum\nolimits_1 \sum_{m=1}^{\infty} v_{n,m} = O(1)$$

as $N \to \infty$, *the summation* $\sum_1$ *being extended over all* n *such that*

$$N^{\frac{1}{2}} < n \leqslant N.$$

LEMMA 3. $$\sum\nolimits_2 \sum_{m=1}^{\infty} v_{n,m} = O(1)$$

as $N \to \infty$, *the summation* $\sum_2$ *being extended over all* n *such that*

$$2\exp\{16(\log N \log\log N)^{\frac{1}{2}}\} < n \leqslant N^{\frac{1}{2}}.$$

LEMMA 4. $$\sum\nolimits_3 \sum_{m=1}^{\infty} v_{n,m} = O\{(\log N \log\log N)^{\frac{1}{2}}\}$$

as $N \to \infty$, *the summation* $\sum_3$ *being extended over all* n *such that*

$$1 \leqslant n \leqslant 2\exp\{16(\log N \log\log N)^{\frac{1}{2}}\}.$$

Of these four results, Lemmas 1 and 4 are obtained by elementary methods, the first by using Cauchy's inequalities for the coefficients of a Taylor series, the second by taking a crude upper bound for the series

† Here, and later, the constants implied in all O's are absolute.

$\sum_{m=1}^{\infty} v_{n,m}$. Lemma 2 depends on a theorem of van der Corput which could, in fact, be used to obtain

$$\bar{q}_n = O\left(\frac{\log N}{\log\log N}\right),$$

and Lemma 3 depends on a modification of a recent theorem of Vinogradow.

8. We consider first Lemmas 1 and 4. Lemma 1 is a particular case of the following result.†

LEMMA 1′. *For any integer $P \geqslant N$ and all $N \geqslant 1$*

$$\left|\sum_{n=P}^{\infty}\sum_{m=1}^{\infty} u_{n,m}\right| \leqslant \frac{AN}{P}\, e^{-P/N}. \tag{8.1}$$

For $N \geqslant 1$, the left side of (8.1) is the modulus of the Nth Taylor coefficient of the function

$$R_P(z) = -\pi \sum_{n=P}^{\infty} \frac{(-1)^{n-1}q^n}{n(1+q^n)},$$

where $q = \exp\{-\pi(1-z)/(1+z)\}$. Let $|z| = \rho = 1-1/N$. Then

$$|q| \leqslant \exp\{-\tfrac{1}{2}\pi(1-\rho)\} \leqslant \exp\left(-\frac{1}{N}\right),$$

so that

$$\begin{aligned} |R_P(z)| &\leqslant \pi \sum_{n=P}^{\infty} \frac{e^{-n/N}}{n(1-e^{-n/N})} \leqslant \frac{\pi e^{-P/N}}{P(1-e^{-P/N})} \sum_{\nu=0}^{\infty} e^{-\nu/N} \\ &\leqslant Ae^{-P/N}P^{-1}(1-e^{-1/N})^{-1} \leqslant ANP^{-1}e^{-P/N}. \end{aligned}$$

The result now follows by Cauchy's inequalities.

Lemma 1 is now obtained by taking $P = N+1$.

LEMMA 4. $$\sum_3 \sum_{m=1}^{\infty} v_{n,m} = O\{(\log N \log\log N)^{\frac{3}{4}}\}$$

as $N \to \infty$, the summation $\sum_3$ being extended over all n such that

$$1 \leqslant n \leqslant 2\exp\{16(\log N \log\log N)^{\frac{3}{4}}\}.$$

For

$$\left|\sum_3 \sum_{m=1}^{\infty} v_{n,m}\right| \leqslant \sum_3 \sum_{m=1}^{\infty} \frac{1}{n^2+m^2} \leqslant \sum_3 \int_0^{\infty} \frac{dx}{n^2+x^2}$$

$$= \tfrac{1}{2}\pi \sum_3 \frac{1}{n} = O\{(\log N \log\log N)^{\frac{3}{4}}\}.$$

† We need the more general result in the proof of (1.10).

9. We now have to abandon elementary methods, and, instead of the series (7.1), we consider the series

$$\sum_n \left| \sum_{m=1}^{\infty} \frac{\exp(2\pi i w_{n,m})}{n^2+(2m-1)^2} \right| = \sum_n \left| \sum_{m=1}^{\infty} v_{n,m} \right|, \tag{9.1}$$

the summation $\sum_n$ being extended over all n in the range

$$2\exp\{16(\log N \operatorname{loglog} N)^{\frac{1}{2}}\} < n \leqslant N.$$

We make use of the following two theorems on the estimation of trigonometrical sums.

THEOREM A. *Suppose that k, L, P are integers, $\frac{1}{2}L \geqslant k \geqslant 2$, $K = 2^k$; that $\phi(x)$ is a real function of x which, together with its derivatives $\phi'(x)$, $\phi''(x)$,..., $\phi^{(k)}(x)$, is defined in the interval $P+1 \leqslant x \leqslant P+L$; and finally that $\phi^{(k)}(x)$ is of constant sign and satisfies*

$$\mu \geqslant |\phi^{(k)}(x)|/k! \geqslant \lambda > 0$$

throughout this interval. Then

$$\max_{1\leqslant L'\leqslant L} \left| \sum_{m=P+1}^{P+L'} e^{2\pi i\phi(m)} \right| \leqslant AM \max\left\{ \left(\frac{\mu^2}{\lambda}\right)^{1/(K-2)}, \left(\frac{1}{\lambda M^k}\right)^{2/K}, \left(\frac{\mu}{\lambda M}\right)^{2/K}, \frac{k}{M} \right\},$$

where M is any number not less than L.

THEOREM B. *Suppose that k, L, M, P are integers, $k \geqslant 7$, $L \geqslant 1$, $M \geqslant 2$; that $\phi(x)$ is a real function of x which, together with its derivatives $\phi'(x)$, $\phi''(x)$,..., $\phi^{(k+1)}(x)$, is defined and continuous in the interval*

$$P+1 \leqslant x \leqslant P+L;$$

that

$$\frac{1}{Q} \leqslant \left| \frac{\phi^{(k+1)}(x)}{(k+1)!} \right| \leqslant \frac{2}{Q} \quad (P+1 \leqslant x \leqslant P+L); \tag{9.2}$$

and finally that

$$Q^{\frac{1}{2}} \leqslant M \leqslant Q, \qquad L \leqslant M.$$

Then

$$\max_{1\leqslant L'\leqslant L} \left| \sum_{m=P+1}^{P+L'} e^{2\pi i\phi(m)} \right| \leqslant Ae^{18k\log^2 k} M^{1-\rho}\log M, \tag{9.3}$$

where

$$\rho = (50k^2\log k)^{-1}.$$

Theorem A is easily deduced from the following theorem of van der Corput.†

† van der Corput (**2**). An earlier version of the theorem [van der Corput (**1**)] deals with differential coefficients in place of differences. A somewhat similar theorem has been proved by Titchmarsh (**13**), but this last form is not immediately applicable here.

THEOREM A′. *If k, L, P are integers, $L > k \geqslant 2$, $K = 2^k$, and if $\phi(m)$ is a real function of m defined for $m = P+1,\ldots, P+L$, and such that*† *(9.4)*

$$\Delta^k\phi(m) \geqslant \sigma > 0 \quad (m = P+1,\ldots, P+L-k),$$

then

$$\left|\sum_{m=P+1}^{P+L} e^{2\pi i\phi(m)}\right| < 25L\max\left\{\left(\frac{\tau^2}{\sigma}\right)^{1/(K-2)}, \left(\frac{1}{\sigma L^k}\right)^{2/K}, \left(\frac{\tau}{\sigma L}\right)^{2/K}\right\},$$

where

$$\tau = \frac{1}{L-k}\{\Delta^{k-1}\phi(P+L-k+1)-\Delta^{k-1}\phi(P+1)\}.$$

For suppose that the conditions of Theorem A are satisfied and that $\phi^{(k)}(x) > 0$ (if not, we work with $-\phi$). Since

$$\Delta^k\phi(x) = \phi^{(k)}(x+\theta k) \quad (0 < \theta < 1),$$

provided that the derivative exists in the interval $(x, x+k)$, the condition (9.4) is clearly satisfied with $\sigma = \lambda k!$. Further, $\phi^{(k-1)}(x)$ increases with x in the interval $(P+1, P+L)$, so that

$$\tau \leqslant \frac{1}{L-k}\{\phi^{(k-1)}(P+L)-\phi^{(k-1)}(P+1)\} = \frac{L-1}{L-k}\phi^{(k)}(P+\xi),$$

where $1 < \xi < L$. Thus $\tau \leqslant 2\mu k!$ (since $L-k \geqslant \frac{1}{2}L$), and from Theorem A′ we have

$$\left|\sum_{m=P+1}^{P+L} e^{2\pi i\phi(m)}\right| < 25L\max\left\{\left(\frac{4\mu^2k!}{\lambda}\right)^{1/(K-2)}, \left(\frac{1}{\lambda k!L^k}\right)^{2/K}, \left(\frac{2\mu}{\lambda L}\right)^{2/K}\right\}$$

$$\leqslant AL\max\left\{\left(\frac{\mu^2}{\lambda}\right)^{1/(K-2)}, \left(\frac{1}{\lambda L^k}\right)^{2/K}, \left(\frac{\mu}{\lambda L}\right)^{2/K}\right\}. \qquad (9.5)$$

For constant μ, λ, and P, the expression on the right of (9.5) is an increasing function of L, so that we may first replace the left side of (9.5) by

$$\max_{2k\leqslant L'\leqslant L}\left|\sum_{m=P+1}^{P+L'} e^{2\pi i\phi(m)}\right|,$$

and then replace L wherever it occurs explicitly on the right side by M. Using the crude inequality

$$\left|\sum_{m=P+1}^{P+L'} e^{2\pi i\phi(m)}\right| \leqslant L'$$

for those values of L' less than $2k$, we obtain Theorem A.

Theorem B is a modification of a recent theorem of Vinogradow.‡ Vinogradow treats k as a constant, and the factor $\exp(18k\log^2 k)$ on the

† Δ^k is the kth difference and is defined by the relations

$$\Delta^k\phi(m) = \Delta\{\Delta^{k-1}\phi(m)\} \quad (k > 1), \qquad \Delta\phi(m) = \Delta^1\phi(m) = \phi(m+1)-\phi(m).$$

‡ Vinogradow (**17**), Theorem 2 b, p. 71.

right side of (9.3) is not determined explicitly.†‡ Since Vinogradow's proof is available only in Russian, it seems worth while to give here the proof of Theorem B in the general case, and this is done in Appendix I.§

It should perhaps be added that the estimate of Lemma 3 can also be obtained by the use of an earlier theorem of Tchudakoff,‖ but this deals with polynomial ϕ and the details become a little more complicated.††

† There are other, but minor, differences. In the original (Vinogradow's) form of the theorem, M was subject to the condition $Q^{k/(2k+1)} \leqslant M \leqslant Q$, and ϕ satisfied (9.2) in the whole interval $(P, P+M)$. I have also extended the range of validity of the theorem from $k \geqslant 12$ to $k \geqslant 7$ (and this is as far as possible), though this is at the expense of the numerical factor in the value of ρ (which is of little interest in our application). The weaker conditions of Theorem B enable us to simplify the discussion of the series (9.1) which follows (and, in particular, to deal only with odd values of k).

‡ As an application of his theorem, Vinogradow proves an estimate for $\sum_{N}^{2N} n^{-s}$ $(s = \sigma+it)$ which at first sight seems to be equivalent to the result

$$\zeta(1+it) = O\{(\log t)^{\frac{2}{3}+\epsilon}\}.$$

The estimate, however, is non-uniform in N and t, and when the factor

$$\exp(18k \log^2 k)$$

is inserted, leads only to the (known) result $\zeta(1+it) = O\{(\log t)^{\frac{2}{3}+\epsilon}\}$.

§ We shorten the proof considerably, however, by appealing to an improved form of Vinogradow's mean-value theorem due to Hua (**5**) which replaces the principal lemma of Vinogradow's proof.

‖ Tchudakoff (**12**), where the theorem is stated without proof. So far as I am aware, no proof has been published. For constant k, Theorem B is an improvement on Tchudakoff's result.

†† The problem of the estimation of the sum (9.1) seems to run parallel with those of the estimations of the order of $\zeta(1+it)$ and of $\pi(x)-\mathrm{li}\,x$. In fact any known estimate of the same type as Theorem B which gives

$$\zeta(1+it) = O\{(\log t)^{\eta+\epsilon}\} \quad (\epsilon > 0)$$

may also be used to prove that

$$\bar{q}_N = O\{(\log N)^{\eta+\epsilon}\},$$

and that

$$\zeta(s) = O\{\exp(\log t)^{\delta}\} \qquad (*)$$

in the region $\sigma \geqslant 1-(\log t)^{-(\eta+\epsilon)}$ $(s = \sigma+it,\ \delta > 0)$.

[See e.g. Titchmarsh (**14**), Tchudakoff (**12**), Flett (**3**).] In this connexion, it should be mentioned that Vinogradow, in an address (**16**), has claimed that Tchudakoff has proved that

$$\pi(x)-\mathrm{li}\,x = O[x \exp\{-(\log x)^{\frac{1}{2}-\epsilon}\}],$$

which is 'equivalent' to (*) with $\eta = \frac{2}{3}$ [though no proof of this result has yet been published. An earlier claim to this result by Vinogradow (**15**) seems doubtful]. For these reasons, a result

$$\bar{q}_N = O\{(\log N)^{\frac{2}{3}+\epsilon}\}$$

seems likely.

10. Before we can apply Theorems A and B, we require some formally obvious facts concerning the successive differential coefficients of $\arctan x$. We have

$$\frac{d^n}{dx^n}(\arctan x) = \frac{(-1)^{n-1}(n-1)!}{(1+x^2)^{\frac{1}{2}n}} \sin\left\{n \arctan \frac{1}{x}\right\},$$

and in particular

$$\frac{d^{2\nu+1}}{dx^{2\nu+1}}(\arctan x) = \frac{(-1)^{\nu+1}(2\nu)!}{(1+x^2)^{\nu+\frac{1}{2}}} \cos\{(2\nu+1)\arctan x\}, \tag{10.1}$$

and

$$\frac{d^{2\nu}}{dx^{2\nu}}(\arctan x) = \frac{(-1)^{\nu}(2\nu-1)!}{(1+x^2)^{\nu}} \sin\{2\nu \arctan x\}. \tag{10.2}$$

I now prove

LEMMA 5. *In each of the intervals*

$$\tan\frac{(r-1)\pi}{2\nu}+\delta \leqslant x \leqslant \tan\frac{r\pi}{2\nu}-\delta \quad (r = 1, 2,..., \nu-1), \tag{10.3}$$

$$\tan\frac{(\nu-1)\pi}{2\nu}+\delta \leqslant x \leqslant \frac{1}{\delta}, \tag{10.4}$$

where ν is any positive integer and $0 < \delta < \frac{1}{2}\nu^{-1}$, $\sin\{2\nu \arctan x\}$ is of constant sign and in absolute value is greater than $\delta(\pi\nu)^{-1}$.

We note first that the condition $\delta < \frac{1}{2}\nu^{-1}$ ensures that the intervals (10.3) and (10.4) do exist (the first interval (10.3) being the smallest).

The points

$$x = 0, \ \tan\frac{\pi}{2\nu}, \ \tan\frac{2\pi}{2\nu},..., \ \tan\frac{(\nu-1)\pi}{2\nu}, \ \infty$$

are the (only) non-negative zeros of $\sin\{2\nu \arctan x\}$. It follows that $\sin\{2\nu \arctan x\}$ is of constant sign in any one of the intervals (10.3), (10.4).

To prove the remainder of the lemma, it is sufficient to show that, when x lies in the rth interval (10.3), the distance of $\arctan x$ from either of the points $(r-1)\pi/(2\nu)$, $r\pi/(2\nu)$ is greater than $\delta/(4\nu^2)$, with a similar result for the interval (10.4), for we shall then have

$$|\sin\{2\nu \arctan x\}| \geqslant \frac{2}{\pi} 2\nu \frac{\delta}{4\nu^2} = \frac{\delta}{\pi\nu}.$$

It is easily seen that the least favourable case is that in which x lies in the interval (10.4), and here we have

$$\begin{aligned}\arctan x - \frac{(\nu-1)\pi}{2\nu} &\geqslant \arctan\left\{\tan\frac{(\nu-1)\pi}{2\nu}+\delta\right\}-\arctan\left\{\tan\frac{(\nu-1)\pi}{2\nu}\right\}\\ &= \arctan\left\{\delta\Big/\left[1+\tan\frac{(\nu-1)\pi}{2\nu}\left(\tan\frac{(\nu-1)\pi}{2\nu}+\delta\right)\right]\right\}\\ &\geqslant \arctan\left(\frac{\delta}{1+\nu(\nu+\delta)}\right) \geqslant \frac{2\delta}{3\{1+\nu(\nu+\frac{1}{2}\nu^{-1})\}} > \frac{\delta}{4\nu^2},\end{aligned}$$

and

$$\tfrac{1}{2}\pi-\arctan x \geqslant \tfrac{1}{2}\pi-\arctan(1/\delta) = \arctan\delta \geqslant \tfrac{2}{3}\delta > \delta/4\nu^2.$$

This completes the proof of the lemma.

We remark also that between any two successive zeros of

$$\sin\{2\nu\arctan x\}$$

lies exactly one zero of $\cos\{(2\nu+1)\arctan x\}$.

11. We return now to the series (9.1). Let k and n be a pair of integers satisfying the conditions:

$$k \text{ even}, \qquad 2 \leqslant k < \log^2 n.$$

With each such pair of integers we associate a set of intervals $J_r = J_r(k,n)$ ($r = 1, 2, \ldots, \frac{1}{2}k$) defined as follows. Let J_r ($r = 1, 2, \ldots, \frac{1}{2}k-1$) denote the interval of values of x such that

$$\tan\frac{(r-1)\pi}{k}+\frac{1}{\log^2 n} \leqslant \frac{2x-1}{n} \leqslant \tan\frac{r\pi}{k}-\frac{1}{\log^2 n},$$

and let $J_{\frac{1}{2}k}$ be the interval of x such that

$$\tan\frac{(\frac{1}{2}k-1)\pi}{k}+\frac{1}{\log^2 n} \leqslant \frac{2x-1}{n} \leqslant \log^2 n.$$

Let $I_r = I_r(k,n)$ ($r = 1, 2, \ldots, \frac{1}{2}k$) denote the set of positive integers m lying in the interval J_r, and denote summation over all m belonging to I_r by $\sum_{I_r}$, over all m belonging to $I_1+I_2+\ldots+I_{\frac{1}{2}k}$ by $\sum_I$, and over all remaining positive integers m by $\sum_R$. Finally let U_r, V_r be respectively the minimum and maximum values of m in I_r.

Applying Lemma 5 (and the remark at the end of § 10) to the function

$$\phi(x) = \frac{N}{\pi}\arctan\left(\frac{2x-1}{n}\right) \quad (x \geqslant 0)$$

[taking $\delta = (\log n)^{-2}$], we obtain

LEMMA 6. *Suppose that n, N are integers,*

$$\phi(x) = \frac{N}{\pi}\arctan\left(\frac{2x-1}{n}\right) \quad (x \geqslant 0),$$

k is an even integer, and $2 \leqslant k < \log^2 n$. Then in each of the intervals $J_r(k,n)$ defined above $\phi^{(k)}(x)$ is of constant sign and

$$\frac{N}{\pi k}\left(\frac{2}{n}\right)^k \frac{2}{\pi k\, 2^{\frac{1}{2}k} \log^{2k+2} n} \leqslant \left|\frac{\phi^{(k)}(x)}{k!}\right| \leqslant \frac{N}{\pi k}\left(\frac{2}{n}\right)^k.$$

Further $|\phi^{(k)}(x)|$ has at most one maximum (and no minima) in each of the intervals $J_r(k,n)$.

We complete our preliminary results by obtaining an estimate for $\sum_R v_{n,m}$. For this we use elementary methods.

LEMMA 7. *If k, n are positive integers, k even and $2 \leqslant k < \log^2 n$, then*

$$|\textstyle\sum_R v_{n,m}| \leqslant \dfrac{Ak}{n \log^2 n},$$

where $\sum_R$ denotes summation over all positive integers m not belonging to one of the sets $I_r(k,n)$.

In any one of the intervals of values of x such that

$$0 < \frac{2x-1}{n} < \frac{1}{\log^2 n},$$

$$\tan\frac{r\pi}{k} - \frac{1}{\log^2 n} < \frac{2x-1}{n} < \tan\frac{r\pi}{k} + \frac{1}{\log^2 n} \quad (r = 1, 2, \ldots, \tfrac{1}{2}k-1),$$

there are at most $O(n/\log^2 n)$ positive integers m, the value of $v_{n,m}$ being $O(1/n^2)$ for such m. Hence the sum of the series $\sum v_{n,m}$ over all such m is $O\{k/(n\log^2 n)\}$. Further, writing $m_0 = \frac{1}{2}n\log^2 n$ we have

$$\left|\sum_{m > m_0 + \frac{1}{2}} v_{n,m}\right| \leqslant \sum_{m > m_0} \frac{1}{m^2} \leqslant \int_{m_0 - 1}^{\infty} \frac{dx}{x^2} \leqslant \frac{A}{n\log^2 n},$$

and the result follows.

12. We now proceed to the proof of Lemma 2 using Theorem A. We may generalize the result of Lemma 2 slightly,† and in fact we have

LEMMA 2′. *Let l, N be positive integers, $l \geqslant 2$. Then, as $N \to \infty$,*

$$\sum_{N^{1/l} < n \leqslant N} \left|\sum_{m=1}^{\infty} v_{n,m}\right| = o(1).$$

Choose k as the least even integer such that $k \geqslant l+2$,‡ and suppose N so large that when $n \geqslant N^{1/(k-2)}$ (and *a fortiori*, when $n > N^{1/l}$)

$$2k \leqslant \log^2 n, \qquad n \geqslant 10^4. \tag{12.1}$$

† There is some interest in doing so since the upper limit of the range of values of n over which Vinogradow's method may be applied (here $N^{\frac{1}{3}}$) is different in the estimates given by various authors.

‡ Alternatively, if we choose $k = [\text{loglog } N]$, we can prove that

$$\bar{q}_N = O(\log N/\text{loglog } N).$$

Then the sets $I_r(k, n)$ defined in § 11 do exist, and by Lemma 6 we may apply Theorem A to the function

$$\phi(x) = \frac{N}{\pi}\arctan\left(\frac{2x-1}{n}\right),$$

taking
$$\mu = \frac{N}{\pi k}\left(\frac{2}{n}\right)^k, \qquad \lambda = \frac{N}{\pi k}\left(\frac{2}{n}\right)^k \frac{2}{\pi k\, 2^{\frac{1}{2}k}\log^{2k+2}n},$$
$$P+1 = U_r, \qquad P+L = V_r, \qquad M = n\log^2 n.$$

It is easily verified that under the conditions (12.1)

$$L = V_r - U_r + 1 \geqslant \frac{n}{k} - \frac{n}{\log^2 n} \geqslant \frac{n}{\log^2 n} \geqslant \log^2 n \geqslant 2k$$

for $r = 1, 2, \ldots, \frac{1}{2}k$, and we therefore have

$$\max_{U_r \leqslant W \leqslant V_r}\left|\sum_{m=U_r}^{W} e^{2\pi i\phi(m)}\right| \leqslant An\log^2 n \max\left(T_1, T_2, T_3, \frac{k}{n\log^2 n}\right),$$

where
$$T_1 = \left(2^{\frac{3}{2}k-1}\frac{N\log^{2k+2}n}{n^k}\right)^{1/(K-2)} \leqslant A\left(\frac{N}{n^k}\right)^{1/(K-2)}\log n,$$

$$T_2 = \left(\frac{\pi^2 k^2}{2^{\frac{1}{2}k+1}}\frac{\log^2 n}{N}\right)^{2/K} \leqslant AN^{-2/K}\log n,$$

$$T_3 = \left(\pi k 2^{\frac{1}{2}k-1}\frac{\log^{2k} n}{n}\right)^{2/K} \leqslant An^{-2/K}\log n.$$

Now, when $N^{1/(k-2)} \leqslant n \leqslant N$,

$$n^{-2/K} \geqslant \max\left\{\left(\frac{N}{n^k}\right)^{1/(K-2)},\ N^{-2/K},\ \frac{k}{n\log^2 n}\right\},$$

so that, for the values of n considered in the lemma,

$$\max_{U_r \leqslant W \leqslant V_r}\left|\sum_{m=U_r}^{W} e^{2\pi i\phi(m)}\right| \leqslant An^{1-2/K}\log^3 n,$$

whence, by partial summation,

$$|\textstyle\sum_{I_r} v_{n,m}| \leqslant An^{-(1+2/K)}\log^3 n. \tag{12.2}$$

From (12.2) and the result of Lemma 7, we therefore have

$$\left|\sum_{m=1}^{\infty} v_{n,m}\right| \leqslant Akn^{-(1+2/K)}\log^3 n + Ak(n\log^2 n)^{-1},$$

and

$$\sum_{N^{1/l} \leqslant n \leqslant N}\left|\sum_{m=1}^{\infty} v_{n,m}\right| \leqslant Ak\left\{\sum_{N^{1/k} < n} n^{-(1+2/K)}\log^3 n + \sum_{N^{1/l} < n}(n\log^2 n)^{-1}\right\}$$
$$= o(1)$$

as $N \to \infty$.

This completes the proof of Lemma 2′, and Lemma 2 is the case $l = 7$.

13. To deal with the remaining terms of the series $\sum\sum v_{n,m}$ we use Theorem B, and prove

LEMMA 3. $$\sum_2 \sum_{m=1}^{\infty} v_{n,m} = O(1)$$

as $N \to \infty$, *the summation* $\sum_2$ *being extended over all* n *such that*

$$2\exp\{16(\log N \log\log N)^{\frac{1}{2}}\} < n \leqslant N^{\frac{1}{2}}. \tag{13.1}$$

I first show that for every n in the range (13.1) and all large N

$$\left|\sum_{m=1}^{\infty} v_{n,m}\right| \leqslant \frac{A}{n}\exp\{-\tfrac{1}{2}(\log N)^{\frac{1}{2}}\} + \frac{A(\log N)^{\frac{1}{2}}}{n\log^2 n}. \tag{13.2}$$

As in § 12, write $$\phi(x) = \frac{N}{\pi}\arctan\left(\frac{2x-1}{n}\right),$$

and let k be an odd integer such that

$$8 \leqslant k+1 < \log^2 n. \tag{13.3}$$

Then (by Lemma 6) in each of the intervals $J_r(k+1, n)$† we have

$$\frac{N}{\pi(k+1)}\left(\frac{2}{n}\right)^{k+1}\{\pi(k+1)2^{\frac{1}{2}(k-1)}\log^{2k+4} n\}^{-1} \leqslant \left|\frac{\phi^{(k+1)}(x)}{(k+1)!}\right| \leqslant \frac{N}{\pi(k+1)}\left(\frac{2}{n}\right)^{k+1},$$

and further $|\phi^{(k+1)}(x)|$ has at most one maximum (and no minima) there. It follows that each interval J_r can be divided into at most

$$2\log\{\pi(k+1)2^{\frac{1}{2}(k-1)}\log^{2k+4} n\}/\log 2 = O(k\log\log n)$$

consecutive pieces in each of which

$$\frac{1}{Q} \leqslant \left|\frac{\phi^{(k+1)}(x)}{(k+1)!}\right| \leqslant \frac{2}{Q},$$

where Q depends on k, n, N, and the particular subdivision, and satisfies

$$\frac{2\pi(k+1)}{N}(\tfrac{1}{2}n)^{k+1} \leqslant Q \leqslant \frac{\pi^2(k+1)^2}{N}(\tfrac{1}{2}n)^{k+1}2^{\frac{1}{2}(k-1)}\log^{2k+4} n. \tag{13.4}$$

Now let‡

$$M = [\tfrac{1}{2}n]+1, \qquad k+1 = 2\left[\frac{1}{2}\frac{\log(NM^3)}{\log\frac{1}{2}n}\right].$$

Then $$M < M^3(\tfrac{1}{2}n)^{-2} \leqslant \frac{1}{N}(\tfrac{1}{2}n)^{k+1} \leqslant M^3. \tag{13.5}$$

It follows that for each sub-interval of $J_r(k+1, n)$ and for large enough N

$$Q^{\frac{1}{3}} \leqslant M \leqslant Q.$$

† We work with $k+1$ in place of k here. Thus all intervals J_r, or sets I_r, are of the form $J_r(k+1, n)$, $I_r(k+1, n)$.

‡ So that (13.3) is satisfied in the range (13.1) (and the intervals $J_r(k+1, n)$ exist) for sufficiently large N. Here $[x]$ denotes the integral part of x.

For the right-hand inequality is obvious from the left-hand inequalities in (13.4) and (13.5). To prove the left-hand inequality it is sufficient [in virtue of the right-hand inequalities in (13.4) and (13.5)] to show that

$$\pi^2(k+1)^2\,2^{\frac{1}{2}(k-1)}\log^{2k+4}n \leqslant M,$$

and this is obvious since the left side is less than

$$\exp(Ak\log\log n) \leqslant \exp\{A(\log N\log\log N)^{\frac{1}{2}}\} \leqslant \tfrac{1}{2}n$$

for large enough N.

Let J'_r denote a typical subdivision of J_r, I'_r the corresponding subset of I_r, and let U'_r, V'_r be the minimum and maximum values of m in I'_r. There are two cases to consider.

(i) Suppose first that $V'_r - U'_r + 1 \leqslant M$. Then we apply Theorem B to the sum $\sum e^{2\pi i\phi(m)}$ taking $P+1 = U'_r$, $P+L = V'_r$, and we obtain immediately

$$\max_{U'_r \leqslant W \leqslant V'_r}\left|\sum_{m=U'_r}^{W} e^{2\pi i\phi(m)}\right| \leqslant Ae^{18k\log^2 k}M^{1-\rho}\log M,$$

where

$$\rho = (50k^2\log k)^{-1},$$

whence, by partial summation,

$$|\textstyle\sum_{I'_r} v_{n,m}| \leqslant \dfrac{A}{n}e^{18k\log^2 k}M^{-\rho}\log M. \tag{13.6}$$

(ii) If, on the other hand, $V'_r - U'_r + 1 > M$, we can subdivide the set I'_r into at most $O(\log^2 n)$ sets of type (i)† for each of which (13.6) is valid.

Thus in either case we have

$$|\textstyle\sum_{I'_r} v_{n,m}| \leqslant \dfrac{A}{n}\log^3 n(\tfrac{1}{2}n)^{-\rho}e^{18k\log^2 k},$$

$$|\textstyle\sum_{I} v_{n,m}| \leqslant \dfrac{Ak^2}{n}\log^3 n\log\log n(\tfrac{1}{2}n)^{-\rho}e^{18k\log^2 k}.$$

Since (always for large enough N)

$$k \leqslant \frac{\log(NM^3)}{\log\frac{1}{2}n} \leqslant \frac{4\log N}{\log\frac{1}{2}n} \leqslant \log N,$$

we have

$$\begin{aligned}(\tfrac{1}{2}n)^{-\rho}e^{18k\log^2 k} &\leqslant \exp\left\{72\frac{\log N}{\log\frac{1}{2}n}(\log\log N)^2 - \frac{1}{800}\frac{\log^3\frac{1}{2}n}{\log^2 N\log\log N}\right\}\\ &\leqslant \exp\left\{\left(\frac{72}{16}-\frac{16^3}{800}\right)(\log N)^{\frac{1}{2}}(\log\log N)^{\frac{1}{2}}\right\}\\ &\leqslant \exp\{-\tfrac{3}{5}(\log N)^{\frac{1}{2}}\}.\end{aligned}$$

† For $m < \frac{1}{2}n\log^2 n + \frac{1}{2}$ in any of the intervals $J_r(k+1, n)$.

Hence

$$|\sum_I v_{n,m}| \leqslant \frac{A}{n}\log^3 N \operatorname{loglog} N \exp\{-\tfrac{3}{5}(\log N)^{\frac{1}{2}}\}$$

$$\leqslant \frac{A}{n}\exp\{-\tfrac{1}{2}(\log N)^{\frac{1}{2}}\} \tag{13.7}$$

for large enough N.

From (13.7) and Lemma 7 we now have (13.2), and the result of the lemma follows immediately on summing this inequality over the range (13.1).

This completes the proof of Lemma 3 and of the result (1.8).

14. *The order of* $\bar{S}_N$. I now consider the sum

$$\bar{S}_N = \bar{q}_1+\bar{q}_2+\ldots+\bar{q}_N,$$

and I prove that, as $N \to \infty$,

$$\bar{S}_N = O(\log^2 N). \tag{14.1}$$

This is a simple consequence of the formula (1.9) for $\bar{q}_N$ and Lemma 1′. For taking $P = [N\log N]+1$ in Lemma 1′ we have

$$\bar{q}_r = \frac{32}{\pi}\sum_{n\leqslant N\log N}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}\cos\{2r\arctan((2m-1)/n)\}}{n^2+(2m-1)^2}+$$

$$+2i(-1)^r+O\left(\frac{re^{-(N\log N)/r}}{N\log N}\right).$$

Since $$\tfrac{1}{2}+\sum_{r=1}^{N}\cos 2r\theta = \frac{\sin(2N+1)\theta}{2\sin\theta} \quad (\theta \neq 0,\ 2\pi,\ldots)$$

and $$\sin\left(\arctan\frac{2m-1}{n}\right) = (2m-1)/\sqrt{\{n^2+(2m-1)^2\}},$$

$$\bar{S}_N = \frac{16}{\pi}\sum_{n\leqslant N\log N}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}\sin\{(2N+1)\arctan((2m-1)/n)\}}{(2m-1)\sqrt{\{n^2+(2m-1)^2\}}}-$$

$$-\frac{16}{\pi}\sum_{n\leqslant N\log N}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}}{n^2+(2m-1)^2}+i\{1-(-1)^N\}+o(1). \tag{14.2}$$

Hence

$$|\bar{S}_N| \leqslant \frac{16}{\pi}\sum_{n\leqslant N\log N}\sum_{m=1}^{\infty}\frac{1}{(2m-1)\sqrt{\{n^2+(2m-1)^2\}}}+O(1)$$

$$\leqslant A\sum_{n\leqslant N\log N}\frac{\log n}{n}+O(1) \leqslant A\log^2 N,$$

and this is (14.1).

It does not seem possible to improve this result by the deep methods of §§ 7–13. We observe that, if the sums with respect to n on the right side of (14.2) are taken from $n = 1$ to $n = \infty$, the relation (14.2) becomes exact without the $o(1)$ term, the series being interpreted as usual as

$$\sum_{n=1}^{\infty}\Big\{\sum_{m=1}^{\infty}\dots\Big\}.$$

More generally, we may consider the sum

$$\bar{S}_N(\theta) = \bar{q}_1 e^{i\theta}+\bar{q}_2 e^{2i\theta}+\dots+\bar{q}_N e^{Ni\theta}.$$

Here $\bar{S}_N(\theta)$ is $O(N)$, uniformly in θ, for

$$|\bar{S}_N(\theta)| \leqslant \sum_{r=1}^{N}|\bar{q}_r| \leqslant \sqrt{\Big\{N\sum_1^N |\bar{q}_r|^2\Big\}} \leqslant AN$$

since

$$\sum_1^{\infty}|\bar{q}_r|^2\rho^{2r} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\bar{Q}(\rho e^{i\theta})|^2\,d\theta = O\Big(\frac{1}{1-\rho}\Big).$$

More detailed (though no deeper) analysis than that above shows that

(i) if θ is of the form $\pm 2\arctan\{(2m'-1)/n'\}$, the fraction $(2m'-1)/n'$ being in its lowest terms, then

$$\bar{S}_N(\theta) = \frac{(-1)^{n'-1}N\pi}{n'^2+(2m'-1)^2}+O\{(n'+m')\log^2 N\};$$

(ii) if θ is of the form $\pm 2\arctan\{2m'/n'\}$, the fraction $2m'/n'$ being in its lowest terms, then

$$\bar{S}_N(\theta) = O\{(n'+m')\log^2 N\};$$

(iii)
$$\bar{S}_N(\pi) = 2Ni+O(\log^2 N).$$

Appendix I

15. Proof of Theorem B

Theorem B. *Suppose that k, L, M, P are integers, $k \geqslant 7$, $L \geqslant 1$, $M \geqslant 2$; that $\phi(x)$ is a real function of x which, together with its derivatives $\phi'(x),\dots,\phi^{(k+1)}(x)$, is defined and continuous in the interval*

$$P+1 \leqslant x \leqslant P+L;$$

that
$$\frac{1}{Q} \leqslant \Big|\frac{\phi^{(k+1)}(x)}{(k+1)!}\Big| \leqslant \frac{2}{Q} \quad (P+1 \leqslant x \leqslant P+L);$$

and finally that
$$Q^{\frac{1}{2}} \leqslant M \leqslant Q, \qquad L \leqslant M. \tag{15.1}$$

Then
$$\max_{1\leqslant L'\leqslant L}\Big|\sum_{n=P+1}^{P+L'} e^{2\pi i\phi(n)}\Big| \leqslant Ae^{18k\log^2 k}M^{1-\rho}\log M, \tag{15.2}$$

where
$$\rho = (50k^2\log k)^{-1}.$$

Define $\phi^{(k+1)}(x)$ in the intervals $(P, P+1)$, $(P+L, P+M)$ by the relations

$$\phi^{(k+1)}(x) = \phi^{(k+1)}(P+1) \quad (P \leqslant x \leqslant P+L),$$

$$\phi^{(k+1)}(x) = \phi^{(k+1)}(P+L) \quad (P+L \leqslant x \leqslant P+M),$$

and let ϕ be defined by integration of these relations, the constants being so chosen that ϕ and its first k derivatives are continuous in $(P, P+M)$. Then ϕ satisfies

$$\frac{1}{Q} \leqslant \left| \frac{\phi^{(k+1)}(x)}{(k+1)!} \right| \leqslant \frac{2}{Q} \tag{15.3}$$

in $(P, P+M)$. With the definition of ϕ extended in this way, it is enough to prove that

$$\left| \sum_{n=P+1}^{P+M} e^{2\pi i \phi(n)} \right| \leqslant A e^{18k \log^2 k} M^{1-\rho} \log M \tag{15.4}$$

provided that

$$Q^{\frac{1}{2}} \leqslant M \leqslant Q. \tag{15.5}$$

For suppose that (15.4) is true subject to (15.3) and (15.5), and that M now satisfies (15.1). If $Q^{\frac{1}{2}} \leqslant L'$, the conditions (15.3) and (15.5) are satisfied when M is replaced by L', and since $L'^{1-\rho} \log L' \leqslant M^{1-\rho} \log M$, (15.2) follows immediately from (15.4). On the other hand, if $L' < Q^{\frac{1}{2}}$, then

$$\left| \sum_{n=P+1}^{P+L'} e^{2\pi i \phi(n)} \right| \leqslant L' < Q^{\frac{1}{2}} \leqslant M^{\frac{1}{2}} \leqslant M^{1-\rho},$$

and (15.2) again follows.

Since $\phi^{(k+1)}(x)$ is continuous, we may further suppose that $\phi^{(k+1)}(x) > 0$ in $(P, P+M)$.

We now consider the proof of (15.4) subject to (15.3) and (15.5). We have first the following lemmas.

LEMMA A.† *Let P and Y be integers, $Y > 1$, $R \geqslant 2c \geqslant 2$, and let $\psi(y)$ be a real function of y defined for $y = P, P+1, \ldots, P+Y-1$ and such that*

$$\frac{1}{R} \leqslant \psi(y+1) - \psi(y) \leqslant \frac{c}{R} \quad (y = P, \ldots, P+Y-2).$$

Further, let $W \geqslant 1$, and let H be the number of values of y for which

$$||\psi(y)|| \leqslant WR^{-1},$$

where $||x||$ denotes the lesser of $x-[x]$, $1+[x]-x$. Then

$$H < (YcR^{-1}+1)(2W+1).$$

† Vinogradow (17), 24, Lemma 9β. I reproduce the proof here for completeness.

For a given real α and integer h there is only one value of y for which

$$\alpha+h < \psi(y) \leqslant \alpha+h+R^{-1}. \tag{15.6}$$

Therefore the number G of values of y for which

$$\alpha < \psi(y) \leqslant \alpha+R^{-1} \pmod 1\dagger$$

is equal to the number of the values of h for which (15.6) is true. Hence $G \leqslant h_2-h_1+1$, where h_2 is the greatest and h_1 the least of these h. But clearly

$$\psi(P) \leqslant \alpha+R^{-1}+h_1, \qquad \alpha+h_2 \leqslant \psi(P+Y-1),$$

whence

$$h_2-h_1+R^{-1} < \psi(P+Y-1)-\psi(P) \leqslant cR^{-1}(Y-1),$$

and

$$G < cR^{-1}(Y-1)+R^{-1}+1 \leqslant YcR^{-1}+1.$$

The result of the lemma now follows from the fact that an interval of length $2WR^{-1}$ may be divided into $[2W+1]$ intervals of length less than R^{-1} ($< \frac{1}{2}$).

LEMMA B. *Let k, l, m, s be positive integers, $k \geqslant 2$, $m \geqslant 2$,*

$$s \geqslant \tfrac{1}{4}k(k+1)+lk,$$

and let

$$T_1 = \sum_{x=1}^{m} \exp\{2\pi i(\alpha_1 x+\ldots+\alpha_k x^k)\}.$$

Then

$$\int_0^1 \ldots \int_0^1 |T_1|^{2s}\, d\alpha_1 \ldots d\alpha_k \leqslant (7s)^{4sl}(\log m)^{2l} m^{2s-\frac{1}{2}k(k+1)+\delta},$$

where

$$\delta = \tfrac{1}{2}k(k+1)\left(1-\frac{1}{k}\right)^l.$$

This is Hua's form of Vinogradow's mean-value theorem.‡

Turning now to the proof of the main theorem, let

$$\sigma = (5k^2 \log k)^{-1}, \qquad m = [Q^{(1-\sigma)(k+1)}]+1$$

(so that $2 \leqslant m \leqslant M$), and write

$$S = \sum_{n=P+1}^{P+M} e^{2\pi i\phi(n)},$$

$$T = T(y) = \sum_{x=1}^{m} e^{2\pi i\{\phi(x+y)-\phi(y)\}} \quad (P \leqslant y \leqslant P+M-m).$$

Then

$$|S| \leqslant m^{-1} \sum_{y=P}^{P+M-m} |T(y)|+m. \tag{15.7}$$

Let

$$T_1 = \sum_{x=1}^{m} e^{2\pi i(\alpha_1 x+\ldots+\alpha_k x^k)}.$$

† We say that $a < b \leqslant c \pmod 1$ if the inequalities are true when a, b, c are replaced by their residues to modulus 1.

‡ Hua (5), Theorem 1.

For $1 \leqslant x \leqslant m$ we have

$$\phi(x+y)-\phi(y) = A_1 x+A_2 x^2+\ldots+A_k x^k+\frac{2\theta}{Q}m^{k+1},$$

where $A_r = A_r(y) = \phi^{(r)}(y)/r!$ and $0 < \theta < 1$. If then the point $(\alpha_1,\ldots,\alpha_k)$ lies in the k-dimensional region Ω_y defined by the inequalities

$$|\alpha_r - A_r| \leqslant \tfrac{1}{2}m^{-r}\left(\frac{m^{k+1}}{Q}\right) \quad (r = 1, 2,\ldots, k), \tag{15.8}$$

for $1 \leqslant x \leqslant m$ we have

$$\begin{aligned}|\phi(x+y)-\phi(y)-(\alpha_1 x+\ldots+\alpha_k x^k)| &\leqslant \sum_{r=1}^{k}|\alpha_r - A_r|x^r+\frac{2m^{k+1}}{Q}\\ &\leqslant \frac{1}{2}\frac{m^{k+1}}{Q}\sum_{r=1}^{k}\left(\frac{x}{m}\right)^r+\frac{2m^{k+1}}{Q}\\ &\leqslant \frac{m^{k+1}}{Q}(\tfrac{1}{2}k+2) \leqslant \frac{km^{k+1}}{Q},\end{aligned}$$

and

$$\begin{aligned}|e^{2\pi i\{\phi(x+y)-\phi(y)\}}-e^{2\pi i(\alpha_1 x+\ldots+\alpha_k x^k)}| &\leqslant |e^{2\pi i k m^{k+1}/Q}-1|\\ &\leqslant \frac{2\pi k m^{k+1}}{Q}.\end{aligned}$$

Hence

$$|T(y)| \leqslant |T_1|+\frac{2\pi k m^{k+2}}{Q},$$

$$|T(y)|^{2s} \leqslant 2^{2s-1}\left\{|T_1|^{2s}+\left(\frac{2\pi k m^{k+2}}{Q}\right)^{2s}\right\}. \tag{15.9}$$

Multiplying both sides of (15.9) by

$$m^{\frac{1}{2}k(k+1)}\left(\frac{Q}{m^{k+1}}\right)^k$$

and integrating over the region Ω_y, we therefore have

$$|T(y)|^{2s} \leqslant 2^{2s}\left\{m^{\frac{1}{2}k(k+1)}\left(\frac{Q}{m^{k+1}}\right)^k\int_{\Omega_y}\ldots\int|T_1|^{2s}\,d\alpha_1\ldots d\alpha_k+\left(\frac{2\pi k m^{k+2}}{Q}\right)^{2s}\right\}. \tag{15.10}$$

The integral

$$\int_{\Omega_y}\ldots\int|T_1|^{2s}\,d\alpha_1\ldots d\alpha_k$$

is equal to the integral of $|T_1|^{2s}$ taken over the region consisting of the points $(\alpha_1-\gamma_1,\ldots,\ \alpha_k-\gamma_k)$, where $\alpha_1,\ldots,\ \alpha_k$ satisfy the conditions (15.8) and $\gamma_1,\ldots,\ \gamma_k$ are (fixed) integers. We say that this region is 'congruent (mod 1) to Ω_y'. Now let y, y' be two integers in the range

$$P \leqslant y \leqslant P+M-m,$$

and let Ω_y, $\Omega_{y'}$ be the corresponding regions defined by (15.8). A necessary condition that Ω_y should intersect with any region congruent (mod 1) to $\Omega_{y'}$† is that

$$||A_k(y)-A_k(y')|| \leqslant m^{-k}\left(\frac{m^{k+1}}{Q}\right). \tag{15.11}$$

Let $\psi(y) = A_k(y)-A_k(y')$. Then

$$\psi(y+1)-\psi(y) = \frac{1}{k!}\{\phi^{(k)}(y+1)-\phi^{(k)}(y)\} = (k+1)\frac{\phi^{(k+1)}(y+\theta)}{(k+1)!},$$

where $0 < \theta < 1$. The conditions of Lemma A are therefore satisfied with $c = 2$, $R = Q/(k+1)$. Taking $W = m/(k+1)$, we see that the number H of y in $P \leqslant y \leqslant P+M-m$ for which (15.11) is possible satisfies

$$H \leqslant \left(\frac{2M(k+1)}{Q}+1\right)\left(\frac{2m}{k+1}+1\right) \leqslant 3km.$$

Since this is independent of y', it follows that

$$\sum_{y=P}^{P+M-m}\int_{\Omega_y}\dots\int |T_1|^{2s}\,d\alpha_1\dots d\alpha_k \leqslant 3km\int_0^1\dots\int_0^1 |T_1|^{2s}\,d\alpha_1\dots d\alpha_k. \tag{15.12}$$

Now suppose that $s \geqslant \frac{1}{4}k(k+1)+lk$.

Then from Lemma B we have

$$\begin{aligned}\int_0^1\dots\int_0^1 |T_1|^{2s}\,d\alpha_1\dots d\alpha_k &\leqslant (7s)^{4sl}(\log m)^{2l}m^{2s-\frac{1}{4}k(k+1)+\delta}\\ &\leqslant (7s)^{4sl}(\log M)^{2l}m^{2s-\frac{1}{4}k(k+1)+\delta}, \end{aligned}\tag{15.13}$$

where $$\delta = \tfrac{1}{2}k(k+1)\left(1-\frac{1}{k}\right)^l.$$

Combining (15.7), (15.9), (15.12), (15.13) we obtain

$$\begin{aligned}|S|^{2s} &\leqslant 2^{2s-1}\left(m^{-2s}M^{2s-1}\sum_{y=P}^{P+M-m}|T(y)|^{2s}+m^{2s}\right)\\ &\leqslant 2^{4s}\left\{M^{2s-1}3k(7s)^{4ls}\left(\frac{Q}{m^{k+1}}\right)^k(\log M)^{2l}m^{1+\delta}+\right.\\ &\qquad\left.+M^{2s}m^{-2s}\left(\frac{2\pi km^{k+2}}{Q}\right)^{2s}+m^{2s}\right\}.\end{aligned}$$

Now $$M^{-5\sigma} \leqslant Q^{-\sigma} \leqslant m^{k+1}Q^{-1} \leqslant 2^{k+1}Q^{-\sigma} \leqslant 2^{k+1}M^{-\sigma},$$

and $$m \leqslant 2Q^{(1-\sigma)/(k+1)} \leqslant 2M^{5/(k+1)}.$$

Further, we choose

$$l = [\tfrac{5}{2}k\log k]+1, \qquad s = [\tfrac{1}{4}k(k+1)+\tfrac{5}{2}k^2\log k+k]+1,$$

† The length of each side of Ω_y is less than 1.

so that $$5(1+\delta) \leqslant 5(1+1/\sqrt{k}) \leqslant 7.$$

We have then

$$|S|^{2s} \leqslant A^{2s}k(7s)^{4sl}(\log M)^{2s}\{M^{2s-1+5\sigma+7/(k+1)}+M^{2s(1-\sigma)}\}.$$

Since $$2s-1+5\sigma+\frac{7}{k+1} \leqslant 2s-\frac{1}{9},$$

$$18s \leqslant 50k^2\log k = \rho^{-1}, \qquad (7s)^{2l} \leqslant e^{18k\log^2 k},$$

this gives

$$\begin{aligned}|S|^{2s} &\leqslant k(Ae^{18k\log^2 k}\log M)^{2s}\{M^{2s(1-\rho)}+M^{2s(1-\sigma)}\}\\ &\leqslant k\{Ae^{18k\log^2 k}M^{1-\rho}\log M\}^{2s},\end{aligned}$$

and so $$|S| \leqslant Ae^{18k\log^2 k}M^{1-\rho}\log M.$$

This completes the proof of the theorem.

Appendix II

16. Additional Note on Hecke's zeta function†

If we omit the factor $(-1)^{n-1}$ in the series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}\cos\{2N\arctan((2m-1)/n)\}}{n^2+(2m-1)^2} \tag{16.1}$$

which occurs in the expression (1.9) for $\bar{q}_N$, we obtain a series which is very similar to that representing Hecke's zeta function $\zeta(s;\lambda^{\frac{1}{4}N})$ in the algebraic field $k(i)$. Hecke defines this function for $\mathscr{R}s > 1$ and $k = \pm 1, \pm 2, \ldots$ by the series

$$\zeta(s;\lambda^k) = \frac{1}{4}\sum_{\mathbf{n}}\frac{e^{4ki\arg \mathbf{n}}}{|\mathbf{n}|^{2s}}, \tag{16.2}$$

where $\mathbf{n}$ runs through all the non-zero integers of the field $k(i)$‡ (i.e. through the set of Gaussian integers). Writing $\mathbf{n} = n+mi$, we therefore have

$$\begin{aligned}\zeta(s;\lambda^k) &= \frac{1}{4}\sum_{m,n=-\infty}^{\infty}\frac{\exp\{4ki\arctan(m/n)\}}{(m^2+n^2)^s}\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\{4k\arctan(m/n)\}}{(m^2+n^2)^s}+\sum_{n=1}^{\infty}\frac{1}{n^{2s}}.\end{aligned} \tag{16.3}$$

† I am indebted to Mr. C. B. Haselgrove for information regarding the contents of this section. In particular, the formula (16.4) has been taken from an unpublished manuscript which he has kindly allowed me to see.

‡ Hecke (4), § 9.

For $s = 1$ this series is of the same form [but for the factor $(-1)^{n-1}$] as the series (16.1) above, and in fact, we can show that *the series* (16.3) *is for* $s = 1$ *effectively the* $2k$*-th Taylor coefficient of a function similar in general form to* $\bar{Q}(z)$.

For consider the function

$$F(z) = \frac{\pi^2(1+z)}{24(1-z)} + \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{q^n}{n(1-q^n)},$$

where

$$q = \exp\left\{-\frac{2\pi(1-z)}{1+z}\right\}.$$

It is easily shown, by the method of § 3, that the Nth Taylor coefficient of $q^n/(1-q^n)$ is the series

$$\frac{n}{\pi}\sum_{m=-\infty}^{\infty}\frac{\cos\{2N\arctan(m/n)\}}{m^2+n^2},$$

whence the Nth coefficient of $F(z)$ is

$$c_N = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\{2N\arctan(m/n)\}}{m^2+n^2} + \sum_{n=1}^{\infty}\frac{1}{n^2} \quad (N \geqslant 1),$$

where the first series on the right is to be interpreted as $\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\dots\right\}$.

Now the function $\zeta(s;\lambda^k)$ can be continued throughout the s-plane,† and in the strip $\frac{1}{2} < \mathscr{R}s \leqslant 1$ possesses an approximate functional equation similar to that for Riemann's zeta function. In particular, for every integer $X > 1$

$$\zeta(1;\lambda^k) = \sum_{\substack{n^2+m^2\leqslant X^2\\ n\geqslant 1,\, m\geqslant 1}}\frac{\cos\{4k\arctan(m/n)\}}{m^2+n^2} + \sum_{n=1}^{\infty}\frac{1}{n^2} + O\left(\frac{k^2}{X^2}\right). \tag{16.4}$$

Choosing $X = k^2$ and using obvious modifications of Lemmas 1 and 2 we therefore have‡

$$\zeta(1;\lambda^k) = c_{2k} + o(1) \quad (k \to \infty),$$

which proves the statement above.

A proof similar to that of the inequality (1.8) now shows that, as $k \to \infty$,

$$\zeta(1;\lambda^k) = O\{(\log k \log\log k)^{\frac{1}{2}}\}.$$

† If $k = 0$, however, the function defined by (16.2) has a pole at $s = 1$.

‡ This relation, and the similarity of c_N to $\bar{q}_N$, may help to explain the similarity between $\bar{q}_N$ and $\zeta(1+it)$ noted in the footnote to (9).

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POLYNOMIAL EXPRESSIONS FOR LAMÉ FUNCTIONS

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1. Introduction

THE study of Lamé functions is complicated by the fact that they are of eight different species. When the functions are squared, they all appear as polynomials but these polynomials are solutions of a differential equation of the third order. In this paper it is shown that squared Lamé functions and also Lamé products can be expressed in terms of the polynomial solutions of a second-order differential equation. These polynomials are identified with those obtained by Halphen's transformation of Lamé's equation.

Lamé's differential equation in its algebraic form can be written

$$4x(x-1)(x-a)\frac{d^2u}{dx^2}+2\{3x^2-2(a+1)x+a\}\frac{du}{dx}-\{n(n+1)x+h\}u=0, \tag{1.1}$$

and there is no loss of generality in taking the finite singularities as 0, 1, and a. I shall assume that a is real and greater than unity. In the Riemannian form

$$u=P\left\{\begin{array}{cccccc} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & -\frac{1}{2}n & x \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}(n+1) & \end{array}\right\}.$$

For n a positive integer, Lamé functions are $2n+1$ solutions of this equation corresponding to $2n+1$ characteristic values of h, and these solutions are polynomials in x multiplied by $x^{\kappa_1}(x-1)^{\kappa_2}(x-a)^{\kappa_3}$ where each of the indices κ_1, κ_2, κ_3 is zero or $\frac{1}{2}$. The species of a function is $2(\kappa_1+\kappa_2+\kappa_3)+1$; functions of the first and third species exist if n is even, and functions of the second and fourth species if n is odd.

The characteristic values of h are in each case real and distinct, as are the zeros of the polynomials.†

† T. J. Stieltjes, *Acta Math.* 6 (1885), 321.

Lamé functions are the solutions of the integral equations

$$E_n(x) = \lambda \int_0^1 P_n^*\left(\sqrt{\frac{xt}{a}}\right) t^{-\frac{1}{2}}(t-1)^{\kappa_2-\frac{1}{2}}(t-a)^{\kappa_3-\frac{1}{2}}(x-1)^{\kappa_2}(x-a)^{\kappa_3}E_n(t)\,dt, \tag{1.2}$$

where P_n^* is the Legendre polynomial of degree n differentiated $2(\kappa_2+\kappa_3)$ times.† Lamé functions are denoted by the symbol $E_n(x)$ in which the coefficient of the highest power of x in the polynomial is unity, or by $E_n^m(x)$ $(m = 1, 2,..., 2n+1)$, if it is desired to distinguish the $2n+1$ functions of degree n. The corresponding second solution is denoted by $F_n(x)$ and defined by the equation

$$F_n(x) = (n+\tfrac{1}{2})E_n(x) \int_{-\infty}^{x} \frac{dt}{E_n^2(t)t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}(t-a)^{\frac{1}{2}}}. \tag{1.3}$$

$E_n(x)F_n(x)$ is called a Lamé *product* and is used in the solution of the equations of a rotating fluid.‡

2. The differential equation for $E_n^2(x)$

The differential equation satisfied by the squares of the solutions of (1.1) is§

$$x(x-1)(x-a)\frac{d^3X}{dx^3} + \tfrac{3}{2}\{3x^2-2(a+1)x+a\}\frac{d^2X}{dx^2} + \\ + \{(3-n-n^2)x-(h+a+1)\}\frac{dX}{dx} - \tfrac{1}{2}n(n+1)X = 0. \tag{2.1}$$

The complete solution of this equation is

$$X = AE_n^2(x) + BE_n(x)F_n(x) + CF_n^2(x), \tag{2.2}$$

where A, B, C are constants. Differentiating the equation $-(n+1)$ times by Leibniz's theorem and writing $u = D^{-n}X$, we have

$$x(x-1)(x-a)\frac{d^2u}{dx^2} + (\tfrac{1}{2}-n)\{3x^2-2(a+1)x+a\}\frac{du}{dx} - \\ - \{n(1-2n)x+h+n^2(a+1)\}u = 0, \tag{2.3}$$

i.e.
$$u = P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & -2n & x \\ n+\frac{1}{2} & n+\frac{1}{2} & n+\frac{1}{2} & \frac{1}{2}-n & \end{Bmatrix}. \tag{2.4}$$

† E. T. Whittaker, *Proc. London Math. Soc.* (2) 14 (1915), 260; C. G. Lambe and D. R. Ward, *Quart. J. of Math.* (Oxford), 5 (1934), 81.

‡ P. Humbert, 'Fonctions de Lamé...', *Mém. des Sc. Math.* 10 (Paris, 1926).

§ E. T. Whittaker and G. N. Watson, *Modern Analysis*, 3rd ed. (Cambridge, 1920), § 23.7.

This is a form of Heun's equation,† with polynomial solutions of degree $2n$ corresponding to $2n+1$ characteristic values of h. The polynomial solutions of (2.3) will be denoted by $G_{2n}(x)$, or $G_{2n}(h_\nu, x)$ if it is desired to specify a particular polynomial, and the coefficient of the highest power of x will be taken as unity.

The $2n+1$ polynomials $G_{2n}(x)$ obviously correspond to the $2n+1$ Lamé functions. They may be defined by a succession of polynomials $u_r(x)$ of degree $n+r$ in x given by the relations

$$u_r(x) = \Big[x(x-1)(x-a)\frac{d^2}{dx^2} + (\tfrac{3}{2}-r)\{3x^2-2(a+1)x+a\}\frac{d}{dx} + \\ + \{3(r-1)^2 - n(n+1)\}x - \{h+(r-1)^2(a+1)\}\Big] u_{r-1}(x)$$

for $r = 1, 2, \ldots, n$, where

$$u_0(x) = E_n^2(x), \qquad u_n(x) = G_{2n}(x).$$

Hence
$$E_n^2(x) = \frac{(n)!}{(2n)!}\left(\frac{d}{dx}\right)^n G_{2n}(x). \tag{2.5}$$

In Heun's notation

$$G_{2n}(x) = F(a, h+n^2a+n^2; -2n, \tfrac{1}{2}-n, \tfrac{1}{2}-n, \tfrac{1}{2}-n; x). \tag{2.6}$$

3. Lamé products

The equation (2.1) differentiated n times gives

$$x(x-1)(x-a)\frac{d^2v}{dx^2} + (n+\tfrac{3}{2})\{3x^2-2(a+1)x+a\}\frac{dv}{dx} + \\ + \{(n+1)(2n+3)x - h - (n+1)^2(a+1)\}v = 0, \tag{3.1}$$

where
$$v = \left(\frac{d}{dx}\right)^{n+1} X.$$

Hence, for characteristic values of h, the solution of (3.1) is

$$v = \left(\frac{d}{dx}\right)^{n+1}\{BE_n(x)F_n(x) + CF_n^2(x)\}. \tag{3.2}$$

Now

$$v = P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & 2n+2 & x \\ -n-\frac{1}{2} & -n-\frac{1}{2} & -n-\frac{1}{2} & n+\frac{3}{2} & \end{Bmatrix}$$

$$= \{x(x-1)(x-a)\}^{-n-\frac{1}{2}} P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & -2n & x \\ n+\frac{1}{2} & n+\frac{1}{2} & n+\frac{1}{2} & \frac{1}{2}-n & \end{Bmatrix}.$$

† K. Heun, *Math. Annalen*, 33 (1899), 161 and 180.

Hence, the substitution

$$v = \{x(x-1)(x-a)\}^{-n-\frac{1}{2}}u$$

transforms the equation (3.1) into the equation (2.3), and a solution of (3.1) is

$$v = \{x(x-1)(x-a)\}^{-n-\frac{1}{2}}G_{2n}(x),$$

and, since $x^{\frac{1}{2}}$ is not a factor of $F_n^2(x)$, we have

$$D^{n+1}E_n(x)F_n(x) = \{x(x-1)(x-a)\}^{-n-\frac{1}{2}}G_{2n}(x),$$

i.e.
$$E_n(x)F_n(x) = D^{-n-1}\{x(x-1)(x-a)\}^{-n-\frac{1}{2}}G_{2n}(x).$$

Therefore

$$E_n(x)F_n(x) = C\int_{\infty}^{x}\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}(x-t)^nG_{2n}(t)\,dt. \tag{3.3}$$

It will be seen later that $G_{2n}(0) = (-1)^{2\kappa_1}a^n$ and hence it may be seen that

$$C = \frac{(-1)^{n+2\kappa_1}\Gamma(n+\frac{3}{2})}{a^n\Gamma(\frac{1}{2})\Gamma(n+1)}. \tag{3.4}$$

An added arbitrary polynomial of degree n may be discarded as not having $x^{\frac{1}{2}}$ as a factor, and the contour of integration should be modified, if necessary, to avoid the singularities of the integrand.

4. Halphen's transformation of Lamé's equation

Halphen† uses the expression for $\wp(u)$ in terms of $\wp(\frac{1}{2}u)$ to transform Lamé's equation into an equation equivalent to (2.3).

Writing

$$x = \frac{(z^2-a)^2}{4z(z-1)(z-a)}, \tag{4.1}$$

$$\frac{dx}{dz} = 2\left\{\frac{x(x-1)(x-a)}{z(z-1)(z-a)}\right\}^{\frac{1}{2}}, \tag{4.2}$$

we have Lamé's equation (1.1) in the form

$$4x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-a)^{\frac{1}{2}}\frac{d}{dx}\{x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-a)^{\frac{1}{2}}\}\frac{du}{dx} - \{n(n+1)x+h\}u = 0;$$

that is

$$z^{\frac{1}{2}}(z-1)^{\frac{1}{2}}(z-a)^{\frac{1}{2}}\frac{d}{dz}\{z^{\frac{1}{2}}(z-1)^{\frac{1}{2}}(z-a)^{\frac{1}{2}}\}\frac{du}{dz} - \left\{\frac{n(n+1)(z^2-a)^2}{4z(z-1)(z-a)}+h\right\}u = 0.$$

Let
$$u = \{z(z-1)(z-a)\}^{-\frac{1}{2}n}v,$$

then

$$z(z-1)(z-a)\frac{d^2v}{dz^2}+(\tfrac{1}{2}-n)\{3z^2-2(a+1)z+a\}\frac{dv}{dz} - \\ -\{2n(\tfrac{1}{2}-n)z+h+n^2a+n^2\}v = 0,$$

† G. H. Halphen, *Traité des Fonctions Elliptiques* (Paris, 1888), t. 2, 471.

and this is the equation (2.3) with the characteristic solutions $v = G_{2n}(z)$.

Hence
$$E_n(x) = 2^{-n}\{z(z-1)(z-a)\}^{-\frac{1}{4}n}G_{2n}(z), \tag{4.3}$$

where
$$x = \frac{(z^2-a)^2}{4z(z-1)(z-a)},$$

i.e.
$$z = \{x^{\frac{1}{2}} \pm (x-1)^{\frac{1}{2}}\}\{x^{\frac{1}{2}} \pm (x-a)^{\frac{1}{2}}\}. \tag{4.4}$$

Hence, if the roots of the Lamé functions be α_r, that is
$$E_n(x) = x^{\kappa_1}(x-1)^{\kappa_2}(x-a)^{\kappa_3}\prod_r (x-\alpha_r),$$

then
$$G_{2n}(z) = (z^2-a)^{2\kappa_1}(z^2-2z+a)^{2\kappa_2}(z^2-2az+a)^{2\kappa_3}\times$$
$$\times \prod_r \{(z^2-a)^2-4\alpha_r z(z-1)(z-a)\}. \tag{4.5}$$

From (4.4) z is real only if $x \leqslant 0$ or $\geqslant a$. The roots of $G_{2n}(z) = 0$ are given by the expression (4.4) with $x = \alpha_r$ where $0 < \alpha_r < a$, or possibly with $x = 0$, 1, or a. Hence, the only real roots of $G_{2n}(z) = 0$ are given by
$$(z^2-a)^{2\kappa_1}(z^2-2az+a)^{2\kappa_3} = 0.$$

Further, the quartic expressions in (4.5) can be written as
$$[z^2-2z\{\alpha_r+(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}\}+a][z^2-2z\{\alpha_r-(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}\}+a]$$
or
$$[z^2-2(z-a)\{\alpha_r+\alpha_r^{\frac{1}{2}}(\alpha_r-1)^{\frac{1}{2}}\}-a][z^2-2(z-a)\{\alpha_r-\alpha_r^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}\}-a].$$

Hence $G_{2n}(z)$ can always be expressed as a product of real quadratic factors.

From (4.5) we have
$$G_{2n}(0) = (-1)^{2\kappa_1}a^n, \qquad G_{2n}(1) = (-1)^{2\kappa_2}(1-a)^n,$$
$$G_{2n}(a) = (-1)^{2\kappa_3}(a^2-a)^n.$$

5. Recurrence relations

Writing
$$G_{2n}(x) = \sum_{r=0}^{2n} a_r a^{n-r}x^r$$

and substituting in (2.3), we obtain the recurrence relations for the coefficients in the form
$$(r+1)(r+\tfrac{1}{2}-n)a_{r+1}-\{h+(n-r)^2(a+1)\}a_r+$$
$$+a(2n+1-r)(n+\tfrac{1}{2}-r)a_{r-1} = 0 \tag{5.1}$$

for $r = 0, 1, 2, \ldots, 2n$. Taking $a_0 = (-1)^{2\kappa_1}$, a_r is a polynomial of degree r in h, and the characteristic equation is
$$a_{2n+1}(h) = 0. \tag{5.2}$$

This can also be expressed as a determinant by eliminating the coefficients a_0 to a_{2n} in the $2n+1$ equations (5.1).

Writing
$$G_{2n}(x) = \sum_{r=0}^{2n} b_r x^{2n-r}$$
and substituting in (2.3) gives the recurrence relations

$$(r+1)(r+\tfrac{1}{2}-n)b_{r+1} - \{h+(n-r)^2(a+1)\}b_r + \\ +a(2n+1-r)(n+\tfrac{1}{2}-r)b_{r-1} = 0 \qquad (5.3)$$

and, taking $b_0 = 1$, we have $a_r = (-1)^{2\kappa_1}b_r$. Hence

$$G_{2n}(x) = \sum_{r=0}^{n-1} a_r\{(-1)^{2\kappa_1}x^{2n-r} + a^{n-r}x^r\} + a_n x^n. \qquad (5.4)$$

If $\kappa_1 = \frac{1}{2}$, $E_n^2(x)$ has no absolute term and therefore in $G_{2n}(x)$ the coefficient of x^n is zero: that is

$$a_n(h) = 0.$$

If $\kappa_1 = 0$,
$$E_n^2(x) = c_0 + (h/a)c_0 x + \ldots.$$

Therefore in $G_{2n}(x)$ we have

$$(n+1)a_{n+1}(h) = ha_n(h),$$

or
$$a(n+1)a_{n-1}(h) = ha_n(h).$$

The characteristic equation (5.2) may therefore be written as a product of two factors, that is

$$a_n(h)\{ha_n(h) - a(n+1)a_{n-1}(h)\} = 0. \qquad (5.5)$$

From the above and from (5.4) it follows that

$$G_{2n}(a/x) = (-1)^{2\kappa_1}a^n x^{-2n}G_{2n}(x). \qquad (5.6)$$

6. Integral equations

Since the $2n+1$ characteristic values of h (being those appropriate to Lamé functions) are distinct and $G_{2n}(x)$ is a Heun polynomial, we have from Lambe and Ward (loc. cit.) §§ 2.32 and 2.33 that, if h_μ and h_ν be two characteristic values of h,

$$\int_C G_{2n}(h_\mu, t)G_{2n}(h_\nu, t)\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}\,dt = 0, \qquad (6.1)$$

where C is a closed curve encircling any two of the singularities 0, 1, and a, and this expression is not zero if $\mu = \nu$.

We have also from the same reference that

$$\sum_{\nu=1}^{2n+1} G_{2n}(h_\nu, x)G_{2n}(h_\nu, t)\{a'_{2n+1}(h_\nu)\}^{-1} = 2a^{2n+1}(n+\tfrac{1}{2})^2(1-xt/a)^{2n}. \qquad (6.2)$$

Differentiating this equation n times with respect to x and with respect to t gives

$$\sum_{\nu=1}^{2n+1} E_n^2(h_\nu, x)E_n^2(h_\nu, t)\{a'_{2n+1}(h_\nu)\}^{-1} = cP_n(1-2xt/a), \qquad (6.3)$$

where c is a constant and P_n is a Legendre function.

Multiplying the equation (6.2) by

$$G_{2n}(h_\nu, t)\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}$$

and integrating round the contour C, we have the integral equation

$$G_{2n}(x) = \lambda \int_C (1-xt/a)^{2n}\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}G_{2n}(t)\,dt, \qquad (6.4)$$

whose only solutions are the polynomials $G_{2n}(x)$. Hence

$$E_n^2(x) = \lambda \int_C (1-xt/a)^n(-t/a)^n\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}G_{2n}(t)\,dt. \qquad (6.5)$$

Also, from (6.2),

$$\sum_{\nu=1}^{2n+1} G_{2n}(h_\nu, a/x)G_{2n}(h_\nu, t)\{a'_{2n+1}(h_\nu)\}^{-1} = 2a^{2n+1}(n+\tfrac{1}{2})^2x^{-2n}(x-t)^{2n};$$

that is, from (5.6),

$$\sum_{\nu=1}^{2n+1} (-1)^{2\kappa_1}G_{2n}(h_\nu, x)G_{2n}(h_\nu, t)\{a'_{2n+1}(h_\nu)\}^{-1} = 2a^{n+1}(n+\tfrac{1}{2})^2(x-t)^{2n}.$$

This leads to the integral equation

$$G_{2n}(x) = \lambda \int_C (x-t)^{2n}\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}G_{2n}(t)\,dt \qquad (6.6)$$

and

$$E_n^2(x) = \lambda \int_C (x-t)^n\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}G_{2n}(t)\,dt. \qquad (6.7)$$

ON SOME DUAL INTEGRAL EQUATIONS

By C. J. TRANTER (*Shrivenham*)

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Summary

A FORMAL solution of the dual integral equations

$$\left.\begin{aligned}\int_0^\infty t\phi(t)J_\nu(\rho t)\,dt &= f(\rho) \quad (0<\rho<1)\\ \int_0^\infty \phi(t)J_\nu(\rho t)\,dt &= F(\rho) \quad (\rho>1)\end{aligned}\right\}$$

is given when $f(\rho)$, $F(\rho)$ are prescribed functions of ρ and $\phi(t)$ is a function of t to be found. Previous work has treated only the case in which $F(\rho)=0$. A solution for the similar pair

$$\left.\begin{aligned}\int_0^\infty \phi(t)J_\nu(\rho t)\,dt &= G(\rho) \quad (0<\rho<1)\\ \int_0^\infty t\phi(t)J_\nu(\rho t)\,dt &= g(\rho) \quad (\rho>1)\end{aligned}\right\}$$

is also given.

1. Introduction

The solution of the dual integral equations

$$\left.\begin{aligned}\int_0^\infty t\phi(t)J_\nu(\rho t)\,dt &= f(\rho) \quad (0<\rho<1)\\ \int_0^\infty \phi(t)J_\nu(\rho t)\,dt &= F(\rho) \quad (\rho>1)\end{aligned}\right\}, \tag{1}$$

where $f(\rho)$, $F(\rho)$ are prescribed functions of ρ and $\phi(t)$ is a function of t to be found, is known for the case in which $F(\rho)=0$. It has been given by King (**1**) and Titchmarsh (**2**). The latter uses Mellin transforms and his solution applies to the more general case in which t is replaced by t^α in the first of equations (1). The validity of his analysis has been discussed by Busbridge (**3**).

The problem of finding a function V which satisfies Laplace's equation in the semi-infinite solid $z>0$ and is such that, on the surface $z=0$, $\partial V/\partial z = f(\rho)$ inside the circle $\rho=1$ and $V=F(\rho)$ outside this circle has been proposed by Nicholson (**4**), who failed to find a solution by

the usual Fourier–Bessel analysis. When cylindrical coordinates (ρ, z) are used, this problem reduces to the solution of equations (1) with $\nu = 0$. I have recently given a solution in this case using Legendre transforms and oblate spheroidal coordinates (**5**). I have now found it possible to modify analysis given by Webster (**6**) to give a formal solution of equations (1) and have thus solved a rather more general problem than that proposed by Nicholson without having recourse to spheroidal coordinates.

It is fairly obvious that a solution of this problem exists only for a limited choice of $f(\rho)$ and $F(\rho)$. That a solution does exist for some $F(\rho) \neq 0$ is clear from the well-known discontinuous integrals of Weber [see Watson (**7***a*)], a particular example being given in § 3. Beyond a brief statement at the end of § 2 of some obviously necessary conditions, the solution of the interesting but seemingly formidable problem of assigning precise conditions to f and F is not attempted.

Similar analysis can be applied to the pair of equations

$$\left.\begin{aligned} \int_0^\infty \phi(t) J_\nu(\rho t)\, dt &= G(\rho) \quad (0 < \rho < 1) \\ \int_0^\infty t\phi(t) J_\nu(\rho t)\, dt &= g(\rho) \quad (\rho > 1) \end{aligned}\right\}. \tag{2}$$

Again, a solution seems only to have been given so far for the case $g(\rho) = 0$ [see, for example, King (**8**)].

2. The solution of equations (1)

Applying Hankel's inversion theorem to the second of equations (1), we have

$$t^{-1}\phi(t) = \int_0^1 \rho V(\rho) J_\nu(t\rho)\, d\rho + \int_1^\infty \rho F(\rho) J_\nu(t\rho)\, d\rho, \tag{3}$$

where

$$V(\rho) = \int_0^\infty \phi(t) J_\nu(\rho t)\, dt \quad (0 < \rho < 1). \tag{4}$$

Assume the function of ρ given by

$$\int_0^\infty \phi(t) J_\nu(\rho t)\, dt$$

to be continuous at $\rho = 1$ and write

$$V(\rho) = \rho^\nu \left[F(1) + \int_\rho^1 (s^2 - \rho^2)^{-\frac{1}{2}} \chi(s)\, ds \right]. \tag{5}$$

Hence

$$\int_0^1 \rho V(\rho) J_\nu(t\rho)\, d\rho = F(1) \int_0^1 \rho^{\nu+1} J_\nu(t\rho)\, d\rho + \\ + \int_0^1 \rho^{\nu+1} J_\nu(t\rho)\, d\rho \int_\rho^1 (s^2-\rho^2)^{-\frac{1}{2}} \chi(s)\, ds$$

$$= F(1) t^{-1} J_{\nu+1}(t) + \int_0^1 \chi(s)\, ds \int_0^s \rho^{\nu+1} (s^2-\rho^2)^{-\frac{1}{2}} J_\nu(t\rho)\, d\rho$$

$$= F(1) t^{-1} J_{\nu+1}(t) + \int_0^1 s^{\nu+1} \chi(s)\, ds \int_0^{\frac{1}{2}\pi} \sin^{\nu+1}\theta J_\nu(ts\sin\theta)\, d\theta, \tag{6}$$

assuming $R(\nu) > -1$ to ensure the vanishing of the first integral on the right-hand side at $\rho = 0$. Under this condition, we have [Watson (7 *b*)]

$$\int_0^{\frac{1}{2}\pi} \sin^{\nu+1}\theta J_\nu(ts\sin\theta)\, d\theta = (\pi/2ts)^{\frac{1}{2}} J_{\nu+\frac{1}{2}}(ts),$$

so that (3) and (6) give

$$\phi(t) = H(t) + (\tfrac{1}{2}\pi t)^{\frac{1}{2}} \int_0^1 s^{\nu+\frac{1}{2}} \chi(s) J_{\nu+\frac{1}{2}}(ts)\, ds, \tag{7}$$

where

$$H(t) = F(1) J_{\nu+1}(t) + t \int_1^\infty \rho F(\rho) J_\nu(t\rho)\, d\rho, \tag{8}$$

a known function of t when $F(\rho)$ is a prescribed function of ρ.

Multiplication of the first of (1) by $\rho^{\nu+1}$ and integration with respect to ρ between 0 and ρ gives

$$\int_0^\infty \phi(t) J_{\nu+1}(\rho t)\, dt = \rho^{-\nu-1} \int_0^\rho \rho^{\nu+1} f(\rho)\, d\rho \quad (0 < \rho < 1).$$

Substitution of $\phi(t)$ from (7) leads to

$$\int_0^\infty t^{\frac{1}{2}} J_{\nu+1}(\rho t)\, dt \int_0^1 s^{\nu+\frac{1}{2}} \chi(s) J_{\nu+\frac{1}{2}}(ts)\, ds = P(\rho) \quad (0 < \rho < 1), \tag{9}$$

where

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}} P(\rho) = \rho^{-\nu-1} \int_0^\rho \rho^{\nu+1} f(\rho)\, d\rho - \int_0^\infty H(t) J_{\nu+1}(\rho t)\, dt, \tag{10}$$

so that $P(\rho)$ is a known function of ρ when $F(\rho)$, $f(\rho)$ are prescribed. By interchanging the order of integration, (9) can be written

$$\int_0^1 s^{\nu+\frac{1}{2}} \chi(s)\, ds \int_0^\infty t^{\frac{1}{2}} J_{\nu+1}(\rho t) J_{\nu+\frac{1}{2}}(st)\, dt = P(\rho) \quad (0 < \rho < 1). \tag{11}$$

Now [Watson (7 c)] the integral in t vanishes for $\rho < s$ and is equal to $(2/\pi)^{\frac{1}{2}} s^{\nu+\frac{1}{2}} \rho^{-\nu-1} (\rho^2 - s^2)^{-\frac{1}{2}}$ for $\rho > s$. Thus (11) can be written

$$\int_0^\rho s^{2\nu+1} (\rho^2 - s^2)^{-\frac{1}{2}} \chi(s)\, ds = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \rho^{\nu+1} P(\rho). \tag{12}$$

This is Schlömilch's integral equation, and its solution [Whittaker and Watson (9)] is

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}} s^{2\nu+1} \chi(s) = [\rho^{\nu+1} P(\rho)]_{\rho=0} + s \int_0^s (s^2 - \rho^2)^{-\frac{1}{2}} \frac{d}{d\rho} \{\rho^{\nu+1} P(\rho)\}\, d\rho. \tag{13}$$

From (10),

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}} \rho^{\nu+1} P(\rho) = \int_0^\rho \rho^{\nu+1} f(\rho)\, d\rho - \rho^{\nu+1} \int_0^\infty H(t) J_{\nu+1}(\rho t)\, dt,$$

so that $\rho^{\nu+1} P(\rho)$ vanishes when $\rho = 0$. Since

$$\frac{d}{d\rho} \{\rho^{\nu+1} J_{\nu+1}(\rho t)\} = t \rho^{\nu+1} J_\nu(\rho t),$$

$$\frac{d}{d\rho} \{\rho^{\nu+1} P(\rho)\} = (2/\pi)^{\frac{1}{2}} \rho^{\nu+1} \left[f(\rho) - \int_0^\infty t H(t) J_\nu(\rho t)\, dt \right]. \tag{14}$$

Hence (13) gives

$$\chi(s) = (2/\pi) s^{-2\nu} \int_0^s \rho^{\nu+1} \left[f(\rho) - \int_0^\infty t H(t) J_\nu(\rho t)\, dt \right] (s^2 - \rho^2)^{-\frac{1}{2}}\, d\rho. \tag{15}$$

Thus the solution of the dual integral equations is given by (7) where $H(t)$ is given by (8) and $\chi(s)$ by (15). When $F(\rho) = 0$, $H(t) = 0$ and the solution reduces to that given by King (1).

The analysis given above is purely formal. Assumptions about the nature of f and F have been made. I have assumed that $R(\nu) > -1$, that certain integrals exist, that the order of integration in certain double integrals can be interchanged, and that

$$\int_0^\infty \phi(t) J_\nu(\rho t)\, dt$$

is a continuous function of ρ at $\rho = 1$.

3. An example

As an example take the case of $\nu = 1$ with $f(\rho) = 0$ and $F(\rho) = \rho^{-1}$. Here equation (8) gives

$$\begin{aligned} H(t) &= J_2(t) + t \int_1^\infty J_1(t\rho)\, d\rho \\ &= J_2(t) + J_0(t) = 2t^{-1} J_1(t). \end{aligned} \tag{16}$$

Since $f(\rho) = 0$, substitution in (15) gives

$$\chi(s) = -(4/\pi s^2) \int_0^s \rho^2(s^2-\rho^2)^{-\frac{1}{2}}\, d\rho \int_0^\infty J_1(t)J_1(\rho t)\, dt$$

$$= -(4/\pi s^2) \int_0^\infty J_1(t)\, dt \int_0^s \rho^2(s^2-\rho^2)^{-\frac{1}{2}}J_1(t\rho)\, d\rho,$$

interchanging the order of integration. The integral in ρ is easily evaluated [Watson (**7** *b*)] by writing $\rho = s\sin\theta$, and we have

$$\chi(s) = -2(2/\pi s)^{\frac{1}{2}} \int_0^\infty J_1(t)J_{\frac{1}{2}}(st)t^{-\frac{1}{2}}\, dt. \tag{17}$$

The integral in (17) is required for $s < 1$ and the evaluation of this [by Watson (**7** *c*)] leads to

$$\chi(s) = -(4s/3\pi)\, {}_2F_1(\tfrac{3}{2}, \tfrac{1}{2}; \tfrac{5}{2}; s^2)$$

$$= (2/\pi s)[(1-s^2)^{\frac{1}{2}} - s^{-1}\sin^{-1}s]. \tag{18}$$

Substitution from (16) and (18) in (7) gives

$$\phi(t) - 2t^{-1}J_1(t) = (2t/\pi)^{\frac{1}{2}} \int_0^1 s^{\frac{1}{2}}[(1-s^2)^{\frac{1}{2}} - s^{-1}\sin^{-1}s]J_{\frac{1}{2}}(ts)\, ds.$$

Now $$t\int s^{-\frac{1}{2}}J_{\frac{1}{2}}(ts)\, ds = -s^{-\frac{1}{2}}J_{\frac{1}{2}}(ts) = -(2/\pi t)^{\frac{1}{2}}(\sin ts)/s,$$

and integration by parts gives

$$\phi(t) - 2t^{-1}J_1(t) = t^{-1}\sin t - (4/\pi t) \int_0^1 s(1-s^2)^{-\frac{1}{2}}\sin ts\, ds.$$

By writing $s = \sin\theta$, the integral in s becomes Bessel's integral for $(\frac{1}{2}\pi)J_1(t)$ and hence, finally

$$\phi(t) = t^{-1}\sin t.$$

4. The solution of equations (2)

Hankel's inversion-theorem applied to the second of equations (2) gives

$$\phi(t) = \int_0^1 \rho\sigma(\rho)J_\nu(t\rho)\, d\rho + \int_1^\infty \rho g(\rho)J_\nu(t\rho)\, d\rho, \tag{19}$$

where $$\sigma(\rho) = \int_0^\infty t\phi(t)J_\nu(\rho t)\, dt \quad (0 < \rho < 1). \tag{20}$$

Write $$M(\rho) = \int_\rho^1 \rho^{1-\nu}\sigma(\rho)\, d\rho, \tag{21}$$

so that $M(1) = 0$ and

$$\partial M/\partial\rho = -\rho^{1-\nu}\sigma(\rho). \tag{22}$$

Then,

$$\begin{aligned}\int_0^1 \rho\sigma(\rho)J_\nu(t\rho)\,d\rho &= -\int_0^1 (\partial M/\partial\rho)\rho^\nu J_\nu(t\rho)\,d\rho \\ &= -[M\rho^\nu J_\nu(t\rho)]_0^1 + \int_0^1 M(\partial/\partial\rho)\{\rho^\nu J_\nu(t\rho)\}\,d\rho \\ &= t\int_0^1 M\rho^\nu J_{\nu-1}(t\rho)\,d\rho,\end{aligned} \tag{23}$$

if we assume $R(\nu) > 0$ and remember that $M(1) = 0$.

Writing

$$M(\rho) = \int_\rho^1 s\xi(s)(s^2-\rho^2)^{-\frac{1}{2}}\,ds, \tag{24}$$

substituting in (23), and changing the order of integration, we get

$$\begin{aligned}\int_0^1 \rho\sigma(\rho)J_\nu(t\rho)\,d\rho &= t\int_0^1 s\xi(s)\,ds\int_0^s \rho^\nu(s^2-\rho^2)^{-\frac{1}{2}}J_{\nu-1}(t\rho)\,d\rho \\ &= t\int_0^1 s^{\nu+1}\xi(s)\,ds\int_0^{\frac{1}{2}\pi}\sin^\nu\theta J_{\nu-1}(ts\sin\theta)\,d\theta \\ &= (\tfrac{1}{2}\pi t)^{\frac{1}{2}}\int_0^1 s^{\nu+\frac{1}{2}}\xi(s)J_{\nu-\frac{1}{2}}(ts)\,ds,\end{aligned} \tag{25}$$

when the integral in θ is evaluated [Watson (**7** *b*)]. Substitution in (19) then gives

$$\phi(t) = K(t) + (\tfrac{1}{2}\pi t)^{\frac{1}{2}}\int_0^1 s^{\nu+\frac{1}{2}}\xi(s)J_{\nu-\frac{1}{2}}(ts)\,ds, \tag{26}$$

where

$$K(t) = \int_1^\infty \rho g(\rho)J_\nu(t\rho)\,d\rho, \tag{27}$$

a known function of t when $g(\rho)$ is given.

Substitution for $\phi(t)$ from (26) in the first of (2) gives

$$\int_0^\infty t^{\frac{1}{2}}J_\nu(\rho t)\,dt\int_0^1 s^{\nu+\frac{1}{2}}\xi(s)J_{\nu-\frac{1}{2}}(ts)\,ds = Q(\rho) \quad (0 < \rho < 1), \tag{28}$$

where

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}}Q(\rho) = G(\rho) - \int_0^\infty K(t)J_\nu(\rho t)\,dt. \tag{29}$$

Interchanging the order of integration, we can write (28)

$$\int_0^1 s^{\nu+\frac{1}{2}}\xi(s)\,ds\int_0^1 t^{\frac{1}{2}}J_\nu(\rho t)J_{\nu-\frac{1}{2}}(st)\,dt = Q(\rho) \quad (0 < \rho < 1). \tag{30}$$

Now [Watson (**7** *c*)] the integral in t vanishes for $\rho < s$ and is equal to

$$(2/\pi)^{\frac{1}{2}} s^{\nu-\frac{1}{2}} \rho^{-\nu} (\rho^2 - s^2)^{-\frac{1}{2}}$$

for $\rho > s$. Thus (30) can be written

$$\int_0^\rho s^{2\nu}(\rho^2 - s^2)^{-\frac{1}{2}} \xi(s)\, ds = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \rho^\nu Q(\rho) \quad .$$

$$= S(\rho), \tag{31}$$

where

$$S(\rho) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \rho^\nu Q(\rho). \tag{32}$$

As in (12), the solution of this integral equation is

$$(\tfrac{1}{2}\pi) s^{2\nu} \xi(s) = S(0) + s \int_0^s (s^2 - \rho^2)^{-\frac{1}{2}} S'(\rho)\, d\rho. \tag{33}$$

The solution of equations (2) is therefore given by (26) where $K(t)$ is given by (27), $S(\rho)$ by (32) and (29), and $\xi(s)$ by (33). When $g(\rho) = 0$, $K(t) = 0$ and the solution reduces to that given by King (**8**).

Again the analysis has been formal and conditions on g and G are clearly necessary for the solution to be valid. In the above I have taken $R(\nu) > 0$, but (23) and the derivation of (25) can be modified in the special case $\nu = 0$, and the final solution still applies in this case.

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REDUCED DETERMINANTAL FORMS FOR S-FUNCTIONS

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1. Introduction

It is well known that S-functions, or bi-alternants, can be expressed as determinants having as elements either the elementary symmetric functions a_r or the homogeneous-product sums h_r. Further, a determinantal form in which the elements are S-functions of type $\{m1^n\}$ has been given by D. E. Littlewood [(**1**) 112]. This form is frequently of order less than that of either the h_r-form or the a_r-form, and is described in this paper as the $\{m1^n\}$-*form*, or the *reduced form*.

In this paper I point out a simple derivation of the reduced form from either the h_r or the a_r-determinantal form, and show that certain powers of an S-function can also be expressed as determinants, all the elements being S-functions of equal rank, or as quotients of determinants. Finally, simple formulae are given for

$$D_m\{\lambda\},\ D_{1^n}\{\lambda\},\ D_{m1^n}\{\lambda\}$$

where D_μ is the differential operator [cf. (**2**)] associated with the partition (μ).

2. Derivation of the reduced form

It is known [(**1**) 110] that every S-function of type $\{p1^\alpha\}$ can be written as

$$h_p a_\alpha - h_{p+1} a_{\alpha-1} + \ldots + (-1)^\alpha h_{p+\alpha}$$

or alternatively as

$$h_{p-1} a_{\alpha+1} - h_{p-2} a_{\alpha+2} + \ldots + (-1)^{p-1} a_{p+\alpha}.$$

Hence, if

$$(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_r),$$

then

$$\{\lambda\} = (-1)^{r-1} \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \ldots & h_{\lambda_1+r-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \ldots & h_{\lambda_2+r-2} \\ h_{\lambda_3-2} & h_{\lambda_3-1} & \ldots & h_{\lambda_3+r-3} \\ \cdot & \cdot & \ldots & \cdot \\ h_{\lambda_r-r+1} & h_{\lambda_r-r+2} & \ldots & h_{\lambda} \end{vmatrix} \times$$

$$\times \begin{vmatrix} (-1)^{r-1}a_{r-1} & (-1)^{r-2}a_{r-2} & \ldots & -a_1 & 1 \\ \cdot & \cdot & \ldots & 1 & \cdot \\ a_2 & -a_1 & \ldots & \cdot & \cdot \\ -a_1 & 1 & \ldots & \cdot & \cdot \\ 1 & \cdot & \ldots & \cdot & \cdot \end{vmatrix}$$

$$= (-1)^{r-1}\begin{vmatrix} (-1)^{r-1}\{\lambda_1, 1^{r-1}\} & \ldots & -\{\lambda_1, 1\} & \{\lambda_1\} \\ (-1)^{r-1}\{\lambda_2-1, 1^{r-1}\} & \ldots & -\{\lambda_2-1, 1\} & \{\lambda_2-1\} \\ (-1)^{r-1}\{\lambda_3-2, 1^{r-1}\} & \ldots & -\{\lambda_3-2, 1\} & \{\lambda_3-2\} \\ \cdot & \ldots & \cdot & \cdot \\ (-1)^{r-1}\{\lambda_r-r+1, 1^{r-1}\} & \ldots & -\{\lambda_r-r+1, 1\} & \{\lambda_r-r+1\} \end{vmatrix}. \tag{1}$$

It may well happen that some of the elements in this determinant are zero. Thus, if $\lambda_i-i+\kappa = 0$, then the κth element in the ith row of the h-form of $\{\lambda\}$ is unity, since $\{0\} = 1$, and the $(r-\kappa+1)$th element in the ith row of (1) is also unity. Further, because of the identities

$$h_p - h_{p-1}a_1 + h_{p-2}a_2 - \ldots + (-1)^p a_p = 0,$$

all the remaining elements in the ith row of (1) will be zero. The effect in such a case is to reduce the order of the determinant. For example*

$$\{432^21\} = \begin{vmatrix} h_4 & h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ 1 & h_1 & h_2 & h_3 & h_4 \\ \cdot & 1 & h_1 & h_2 & h_3 \\ \cdot & \cdot & \cdot & 1 & h_1 \end{vmatrix} \times \begin{vmatrix} a_4 & -a_3 & a_2 & -a_1 & 1 \\ -a_3 & a_2 & -a_1 & 1 & \cdot \\ a_2 & -a_1 & 1 & \cdot & \cdot \\ -a_1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} \{41^4\} & -\{41^3\} & \{41^2\} & -\{41\} & \{4\} \\ \{21^4\} & -\{21^3\} & \{21^2\} & -\{21\} & \{2\} \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \{41^4\} & \{41^2\} \\ \{21^4\} & \{21^2\} \end{vmatrix}.$$

Alternatively

$$\{432^21\} = \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ a_3 & a_4 & a_5 & a_6 \\ 1 & a_1 & a_2 & a_3 \\ \cdot & \cdot & 1 & a_1 \end{vmatrix} \times \begin{vmatrix} -h_3 & h_2 & -h_1 & 1 \\ h_2 & -h_1 & 1 & \cdot \\ -h_1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} -\{41^4\} & \{31^4\} & -\{21^4\} & \{1^5\} \\ -\{41^2\} & \{31^2\} & -\{21^2\} & \{1^3\} \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \{41^4\} & \{21^4\} \\ \{41^2\} & \{21^2\} \end{vmatrix}.$$

The columns of (1) which remain in the reduced form correspond to those values of κ for which $\lambda_i-i+\kappa \neq 0$, where $0 < \kappa \leqslant r$. It is instructive to derive these values of κ from the conjugate partition

* This corrects the example given in (1) 112.

$(\tilde{\lambda}) = (\mu_1, \mu_2, \ldots, \mu_s)$. Consideration of the graph of (λ) shows that (i) if $\mu_j \geqslant i$, then $\lambda_i \geqslant j$, and (ii) if $\mu_j < i$, then $\lambda_i < j$. In either case

$$\lambda_i - i + \mu_j - j + 1 \neq 0.$$

Hence the required values of κ are given by the positive values of $\mu_j - j + 1$. Thus the reduced $\{m1^n\}$ form is of order l, where l is the number of parts for which $\mu_j > j-1$, or equally the number for which $\lambda_i > i-1$. The number of parts for which $\mu_j \geqslant j$, or equally for which $\lambda_i \geqslant i$, is, however, the rank of the partition: that is, the number of nodes in the leading diagonal of the graph of the partition. It now becomes clear that the Frobenius notation for partitions is very appropriate to the $\{m1^n\}$ form as it gives equal emphasis to rows and columns. It enables the reduced form to be written down at sight.* Thus

$$\{432^21\} = \begin{Bmatrix} 31 \\ 42 \end{Bmatrix} = \begin{vmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 4 \end{Bmatrix} \\ \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \end{vmatrix} = \begin{vmatrix} \{41^4\} & \{21^4\} \\ \{41^2\} & \{21^2\} \end{vmatrix}$$

and, for the conjugate S-function,

$$\{5421\} = \begin{Bmatrix} 42 \\ 31 \end{Bmatrix} = \begin{vmatrix} \begin{Bmatrix} 4 \\ 3 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \\ \begin{Bmatrix} 4 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \end{vmatrix} = \begin{vmatrix} \{51^3\} & \{31^3\} \\ \{51\} & \{31\} \end{vmatrix}.$$

The reduced form is completely specified by its diagonal and so we may write

$$\begin{Bmatrix} p & q & r \ldots \\ a & b & c \ldots \end{Bmatrix} = \begin{vmatrix} \begin{Bmatrix} p \\ a \end{Bmatrix} \begin{Bmatrix} q \\ b \end{Bmatrix} \begin{Bmatrix} r \\ c \end{Bmatrix} \ldots \end{vmatrix}.$$

The diagonal terms, in fact, give the Γ-shaped partition graphs (including possibly a row or a column as a special case) which, when applied intact in succession each into the angle of the previous one, produce the graph of the given partition (λ), just as the diagonal terms in the h-form give the successive rows of the graph and the diagonal terms in the a-form give the successive columns. Hence the $\{m1^n\}$ form may be written down at sight either from the graph or from the Frobenius form of the partition.

Since $\{\lambda\}$ occurs in the product of the elements in the leading diagonal of the reduced form and in no other term in the expansion of the determinant, the sign to be attached to the reduced form as given above is always positive.

* This is equivalent to Theorem XII in (1) 112 but is more simply expressed.

3. Derivation of various hybrid forms

The various forms given by Littlewood [(**1**) 114], in which some columns have elements of type h_r and the remaining columns have elements of type $\{m1^n\}$, may be obtained by similar methods. For example,

$$\{432^21\} = |h_4h_3h_2h_2h_1| \times \begin{vmatrix} . & . & a_2 & -a_1 & 1 \\ . & a_2 & -a_1 & 1 & . \\ a_2 & -a_1 & 1 & . & . \\ -a_1 & 1 & . & . & . \\ 1 & . & . & . & . \end{vmatrix}$$

$$= \begin{vmatrix} \{61^2\} & \{41^2\} & \{41\} & \{4\} \\ \{41^2\} & \{21^2\} & \{21\} & \{2\} \\ \{21^2\} & . & . & 1 \\ \{1^3\} & . & 1 & . \end{vmatrix}.$$

Similarly by multiplying the a-form by variants of the unit determinant, expressions involving S-functions of type a_r and $\{m1^n\}$ may be obtained.

4. Determinant expressions for powers of an S-function

The $\{m1^n\}$-form differs from the h-form and the a-form in that all its minors are S-functions. Thus a minor

$$\begin{vmatrix} \{p\} & \{q\} & \cdots \\ \{\alpha\} & \{\beta\} & \cdots \end{vmatrix}$$

of order l_1 is, with the appropriate sign attached, equal to the corresponding S-function of rank l_1. The various Laplace expansions thus yield a variety of expressions for a given S-function of rank l, each of the form

$$\{\nu_1\}\{\eta_1\} - \{\nu_2\}\{\eta_2\} + \{\nu_3\}\{\eta_3\} - \cdots \pm \{\nu_x\}\{\eta_x\}$$

where $\{\nu_i\}$ is of rank l_1, $\{\eta_i\}$ is of rank l_2, and $l_1 + l_2 = l$.

Furthermore the various compound determinants derived from the $\{m1^n\}$-form have elements which are single S-functions, all those in the κth compound being of rank κ. Hence, using Sylvester's theorem [(**3**) 87] that the κth compound of a determinant of order m is equal to the ${}^{m-1}C_{\kappa-1}$th power of the determinant, we have a means of expressing the ${}^{m-1}C_{\kappa-1}$th powers of an S-function of rank m as a determinant of order ${}^mC_\kappa$, in which every element is of rank κ. For any given power there

may be two such determinants, correspondingto κ and $m-\kappa+1$. For example,

$$\{654^2 1^2\} = \begin{vmatrix} \{5310\} \\ \{5210\} \end{vmatrix} = \begin{vmatrix} \{5\}\{3\}\{1\}\{0\} \\ \{5\}\{2\}\{1\}\{0\} \end{vmatrix},$$

$$\{654^2 1^2\}^3 = \begin{vmatrix} \{310\}\{510\}\{530\}\{531\} \\ \{210\}\{510\}\{520\}\{521\} \end{vmatrix}$$

$$= \begin{vmatrix} \{53\}\{51\}\{50\}\{31\}\{30\}\{10\} \\ \{52\}\{51\}\{50\}\{21\}\{20\}\{10\} \end{vmatrix}.$$

Similarly by using Jacobi's and Franke's theorems [(**3**) 108] on compound determinants we can write down various expressions for certain powers of $\{\lambda\}$ as quotients of determinants which have single S-functions as elements. Typical examples are

$$\{654^2 1\}^2 = \begin{vmatrix} \{510\}\{530\}\{531\} \\ \{510\}\{520\}\{521\} \end{vmatrix} \div \begin{vmatrix} \{5\} \\ \{5\} \end{vmatrix}$$

$$\{654^2 1\} = \begin{vmatrix} \{50\}\{31\}\{30\}\{10\} \\ \{50\}\{21\}\{20\}\{10\} \end{vmatrix} \div \begin{vmatrix} \{10\}\{30\} \\ \{10\}\{20\} \end{vmatrix}.$$

5. Effect of the operators on the reduced form

(i) *The operator* $D_m\{\lambda\}$:

The effect of D_m on $\{p1^a\}$ is to give the term $\{p-m, 1^a\}$ and the term $\{p-m+1, 1^{a-1}\}$. Hence $D_m\{p1^a\} = \{p-m\}\{1^a\}$. Suppose

$$\{\lambda\} = \begin{vmatrix} \{p1^a\} & \{p1^b\} & \{p1^c\} & . & . & . \\ \{q1^a\} & \{q1^b\} & \{q1^c\} & . & . & . \\ \{r1^a\} & \{r1^b\} & \{r1^c\} & . & . & . \\ . & . & . & . & . & . \end{vmatrix}$$

where the determinant is of order l. If $(m_1\, m_2)$ is any two-part partition of m, and ϕ_1, ϕ_2 are any symmetric functions, then

$$D_m \phi_1 \phi_2 = \sum D_{m_1} \phi_1 . D_{m_2} \phi_2$$

summed over all partitions of m into two parts. For the purpose of operating on the products of l terms arising in the expansion of the determinant we regard the operators corresponding to the two-part partitions as being broken up into D_{m_1}, D_{m_2}, D_0, D_0,..., and the summation is taken for all distinct permutations of the operators. Hence $D_m\{\lambda\}$ will contain terms of the type

$$\begin{vmatrix} \{p-m_1\}\{1^a\} & \{p-m_1\}\{1^b\} & \{p-m_1\}\{1^c\} & . & . & . \\ \{q-m_2\}\{1^a\} & \{q-m_2\}\{1^b\} & \{q-m_2\}\{1^c\} & . & . & . \\ \{r1^a\} & \{r1^b\} & \{r1^c\} & . & . & . \\ . & . & . & . & . & . \end{vmatrix},$$

which are clearly zero, and so are all terms obtained by taking partitions of m into three or more non-zero parts. The only non-vanishing terms in $D_m\{\lambda\}$ are those in which m is regarded as partitioned into m and $l-1$ zeros, and these are all included in

$$D_m\{\lambda\} = -\begin{vmatrix} \{p1^a\} & \{p1^b\} & \{p1^c\} & \dots & \{p-m\} \\ \{q1^a\} & \{q1^b\} & \{q1^c\} & \dots & \{q-m\} \\ \{r1^a\} & \{r1^b\} & \{r1^c\} & \dots & \{r-m\} \\ \dots & \dots & \dots & \dots & \dots \\ \{1^a\} & \{1^b\} & \{1^c\} & \dots & 0 \end{vmatrix},$$

so that the result is obtained by bordering the reduced form of $\{\lambda\}$ and changing the sign.

(ii) *The operator* $D_{1^m}\{\lambda\}$:

Since
$$\begin{aligned} D_{1^m}\{p1^a\} &= \{p, 1^{a-m}\}+\{p-1, 1^{a-m+1}\} \\ &= \{p-1\}\{1^{a-m+1}\}, \end{aligned}$$

any decomposition of D_{1^m} into $D_{1^{m_1+1}} D_{1^{m_2-1}} \dots$ leads to terms in $D_{1^m}\{\lambda\}$ of the type

$$\begin{vmatrix} \{p-1\}\{1^{a-m_1}\} & \{p-1\}\{1^{b-m_2}\} & \{p1^c\} & \dots \\ \{q-1\}\{1^{a-m_1}\} & \{q-1\}\{1^{b-m_2}\} & \{q1^c\} & \dots \\ \{r-1\}\{1^{a-m_1}\} & \{r-1\}\{1^{b-m_2}\} & \{r1^c\} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

which vanish. The non-vanishing terms are given by

$$D_{1^m}\{\lambda\} = -\begin{vmatrix} \{p1^a\} & \{p1^b\} & \{p1^c\} & \dots & \{p-1\} \\ \{q1^a\} & \{q1^b\} & \{q1^c\} & \dots & \{q-1\} \\ \{r1^a\} & \{r1^b\} & \{r1^c\} & \dots & \{r-1\} \\ \dots & \dots & \dots & \dots & \dots \\ \{1^{a-m+1}\} & \{1^{b-m+1}\} & \{1^{c-m+1}\} & \dots & 0 \end{vmatrix}.$$

(iii) *The operator* $D_{m1^n}\{\lambda\}$:

We note first that an S-function of rank unity can occur only in products of S-functions of rank unity, and that its coefficient in any such product is unity. Suppose that $\{m1^n\}$ occurs in the product $\{m_1 1^{n_1}\}\{m_2 1^{n_2}\}$. Then $D_{m1^n}\{\lambda\}$ will contain the sum of determinants obtained from the reduced form of $\{\lambda\}$ by operating on two rows with $D_{m_1 1^{n_1}}$ and $D_{m_2 1^{n_2}}$ respectively, leaving the remaining rows unaltered. Since $D_{m_1 1^{n_1}}\{p1^a\} = \{p-m_1\}\{1^{a-n_1}\}$, then $D_{m1^n}\{\lambda\}$ will contain

$$\begin{vmatrix} \{p-m_1\}\{1^{a-n_1}\} & \{p-m_1\}\{1^{b-n_1}\} & \{p-m_1\}\{1^{c-n_1}\} & \dots \\ \{q-m_2\}\{1^{a-n_2}\} & \{q-m_2\}\{1^{b-n_2}\} & \{q-m_2\}\{1^{c-n_2}\} & \dots \\ \{r1^a\} & \{r1^b\} & \{r1^c\} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

which does not vanish unless $n_1 = n_2$. But $\{m1^n\}$ will also occur in the product $\{m_1 1^{n_2}\}\{m_2 1^{n_1}\}$ and so, if $n_1 \neq n_2$, $D_{m1^n}\{\lambda\}$ will also contain a determinant which differs from the above merely by the interchange of n_1 and n_2, and therefore differs from it only in sign.

Similarly, if $\{m1^n\}$ occurs in $\{m_1 1^{n_1}\}\{m_2 1^{n_2}\}\{m_3 1^{n_3}\}$, where $n_1 \neq n_2 \neq n_3$, it will occur, always with the same coefficient, in all six products formed by permuting the suffixes of the n's in all possible ways. The set of six determinants obtained by operating on a fixed set of three rows of $\{\lambda\}$ with the corresponding six triads of operators in turn, leaving the remaining rows unchanged, will therefore vanish in pairs, each even permutation of the suffixes of the n's giving a determinant opposite in sign to one given by an odd permutation.

Proceeding in this way it becomes evident that the only non-vanishing terms of $D_{m1^n}\{\lambda\}$ are those determinants in which a single row is operated upon with D_{m1^n}. These are all included in the bordered determinant

$$D_{m1^n}\{\lambda\} = -\begin{vmatrix} \{p1^a\} & \{p1^b\} & \{p1^c\} & \ldots & \{p-m\} \\ \{q1^a\} & \{q1^b\} & \{q1^c\} & \ldots & \{q-m\} \\ \{r1^a\} & \{r1^b\} & \{r1^c\} & \ldots & \{r-m\} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \{1^{a-n}\} & \{1^{b-n}\} & \{1^{c-n}\} & \ldots & 0 \end{vmatrix}.$$

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NOTE ON A TWO-POINT BOUNDARY PROBLEM (II)

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THE present note is a continuation of a paper with this title† in which the existence of a solution of J. M. Whittaker's general problem has been proved for a *subclass of complete sequences* of integers p_i, q_j with the restriction of boundedness on the sequences of values a_i, b_j. It is assumed that the reader is able to refer to the previous note, the formula-numbering of which is continued into the present paper.

J. M. Whittaker has proved [(**2**) 50–1]‡, for *every pair of complete sequences* of integers p_i, q_j, the existence of an integral function satisfying conditions (2) if the values a_i, b_j satisfy certain conditions of smallness depending only on p_i and q_j. Using the method of the previous note a similar result is obtained here and stated in Theorem 3. The method allows an immediate extension to the case of a finite or enumerable set of boundary points, given in Theorem 4. A slight modification of the method, by the translation $z-\frac{1}{2}$, leads to results relating to complete sequences with 'bounded span'. In this case the information about the order of magnitude of the values a_i, b_j is more precise, and is given in Theorem 5 for the two-point case, and extended to the many-point case in Theorem 6.

The completeness of the sequences p_i, q_j expressed in formulae (3) and (4) can also be demonstrated by properties of the matrix B of equations (7). We consider the square matrix formed from the first m_r rows and m_r columns, e.g. (4), of the matrix (8), which is reduced in B to a matrix of $P(m_r)$ rows and $m_r-Q(m_r) = P(m_r)$ columns, i.e. to a square matrix. Omitting from this matrix the rows and columns which constitute the same type of matrix for m_{r-1}, I denote by D_r the remaining square matrix, which has $P(m_r)-P(m_{r-1})$ rows and columns. It follows from (3) that D_r has no zero elements in its diagonal or to the right of it, and from (4) that all elements below D_r are zero. The matrices D_r are situated along the diagonal of B; their union, with zero elements added in the remaining places, form an infinite matrix which may be called a *diagonal string of square matrices*, and will be denoted by D.

† *Quart. J. of Math.* (Oxford), 19 (1948), 109–16.

‡ References are given in the previous note.

An example for D_r is the matrix

$$\begin{bmatrix} 1/2! & 1/5! & 1/7! & 1/8! \\ 1/1! & 1/4! & 1/6! & 1/7! \\ 0 & 1/2! & 1/4! & 1/5! \\ 0 & 0 & 1/0! & 1/1! \end{bmatrix}$$

the determinant of which can be written in the form

$$\frac{1}{3!6!8!9!} \begin{vmatrix} \{3,1\} & \{6,1\} & \{8,1\} & \{9,1\} \\ \{3,2\} & \{6,2\} & \{8,2\} & \{9,2\} \\ \{3,4\} & \{6,4\} & \{8,4\} & \{9,4\} \\ \{3,8\} & \{6,8\} & \{8,8\} & \{9,8\} \end{vmatrix}$$

where $\{a,b\} = a(a-1)(a-2)...(a-b+1)$. This determinant is positive as has been proved by A. C. Aitken [**(1)** Lemma 2, 459], and a similar result holds for all the matrices D_r. Hence D has a reciprocal D^{-1}, a diagonal string formed from the square matrices D_r^{-1}. The infinite matrix $C = D^{-1}B$ is therefore an upper semi-matrix with all diagonal elements equal to unity. Corresponding to the nth row of C, we consider the function of the real variable t

$$C_n(t) = \sum_{k=0}^{\infty} |c_{n,n+k}| t^k \quad (0 \leqslant t \leqslant 1;\ n = 1, 2,...), \tag{19}$$

where $c_{n,n} = 1$ so that $C_n(0) = 1$. Also, if $||B||$ is the K_r-bound of B, and $||D_r||$ the K_r-bound of the matrix D_r which contains the elements of the nth row, then $C_n(1) \leqslant ||B||\,||D_r||^{-1}$, so that

$$1 < C_n(1) < \infty \quad (n = 1, 2,...). \tag{20}$$

Hence, given a small positive ϵ, there exists a unique sequence t_n such that t_n is the greatest positive number satisfying

$$C_n(t_n) \leqslant 2-\epsilon, \quad 1 \geqslant t_n \geqslant t_{n+1} > 0 \qquad (\epsilon \text{ small and positive}). \tag{21}$$

We then form the sequence

$$u_n = t_1 t_2 ... t_n. \tag{22}$$

If the pair of complete sequences p_i, q_j is given, then D is uniquely determined, and so is the sequence u_n if, in addition, ϵ is fixed. We replace equation $BX = A$ by

$$CX = A', \quad \text{where} \quad A' = D^{-1}A. \tag{23}$$

Substituting $x_n = u_n y_n$, and dividing the nth equation in (23) by u_n, we obtain the equations

$$c_{n,n} y_n + c_{n,n+1} t_{n+1} y_{n+1} + c_{n,n+2} t_{n+1} t_{n+2} y_{n+2} + ... = A'_n/u_n. \tag{24}$$

To every solution Y of (24) corresponds a solution X of (23), given by $x_n = u_n y_n$. Written in matrix form, equations (24) are

$$C'Y = A'', \tag{25}$$

where

$$A''_n = A'_n/u_n, \qquad c'_{nk} = 0 \quad (k < n), \qquad c'_{nn} = 1,$$
$$c'_{nk} = c_{nk} t_{n+1} t_{n+2} \ldots t_{k-n} \quad (k > n).$$

The matrix C' is a K_r-matrix which satisfies the condition $||C' - I|| < 1$, so that (25) has a unique solution in the algebra of K_r-matrices given by $Y = C'^{-1}A''$, provided that

$$A'' \text{ is a } K_r\text{-matrix}, \tag{26}$$

i.e. A''_n is bounded. The corresponding solution of (23), $x_n = u_n y_n$, is therefore bounded, so that X is a K_r-matrix.

Denoting by U the diagonal matrix of diagonal elements u_n, we can write

$$X = UY = U(C'^{-1}A'') = (UC'^{-1})A'',$$

since all three matrices are K_r-matrices. Now $C' = U^{-1}CU$ and hence

$$C'^{-1} = U^{-1}C^{-1}U,$$

and

$$UC'^{-1} = U(U^{-1}C^{-1}U) = (UU^{-1})(C^{-1}U) = C^{-1}U,$$

since all the matrices are column-finite, C^{-1} being the unique two-sided upper semi-reciprocal of the upper semi-matrix C. Thus

$$X = (C^{-1}U)(U^{-1}A') = C^{-1}A',$$

since the existence of the first product implies the existence and equality of the second. The last result shows that the solution of equation (23) is independent of the sequence u_n and therefore is independent of ϵ, and that, if (26) is satisfied, the solution can be obtained in the usual way from (23). Writing (23) in the form $(D^{-1}B)X = D^{-1}A$, we have

$$D\{(D^{-1}B)X\} = D(D^{-1}A). \tag{27}$$

Now B and X are K_r-matrices; hence BX exists, and from its existence it follows that

$$\sum_{i=1}^{M} d_{n,i}^{-1} \sum_{j=1}^{\infty} B_{i,j} x_j = \sum_{j=1}^{\infty} \Big(\sum_{i=1}^{M} d_{n,i}^{-1} B_{i,j} \Big) x_j,$$

i.e. $D^{-1}(BX) = (D^{-1}B)X$. But BX, D, D^{-1} are row-finite; hence the left-hand side of (27) is $D\{D^{-1}(BX)\} = (DD^{-1})(BX) = BX$. On the right-hand side all three matrices are column-finite, giving

$$(DD^{-1})A = A.$$

Hence, if X is a solution of (23), we have $BX = A$, showing that X is the required solution of our problem.

There remains to show that, for any pair of complete sequences p_i, q_j, *infinite* sequences a_i, b_j exist satisfying (26), since it is trivially satisfied if the number of non-zero elements is finite. Let $\delta(a_i)$ be the K_r-bound of the matrix D_r^{-1} which contains elements of the row corresponding to p_i, and $\delta(b_j)$ that corresponding to the column q_j. We can form *non-decreasing* sequences $\phi(a_i)$, $\phi(b_j)$ such that $0 < \delta(m) \leqslant \phi(m)$. Then

$$|a_i| \leqslant Ku_i/\phi(a_i), \qquad |b_j| \leqslant Ku_j/\phi(b_j) \tag{28}$$

will ensure that (26) be satisfied. Thus we have proved:

THEOREM 3. *Sufficient conditions for the existence of an integral function $f(z)$ satisfying* (1) *and* (2) *are:* (i) *the sequences p_i and q_j are complete,* (ii) *the sequences a_i, b_j satisfy* (28). *The function $f(z)$ is then uniquely determined with $|x_k|$ bounded.*

COROLLARY. *When p_i, q_j can be made complete by adding terms to q_j, then giving arbitrary values to the corresponding b_j, we obtain an infinity of solutions.*

We now consider the problem of an enumerable set of points $z_i \neq 0$ ($i = 1, 2, \ldots$), not necessarily on the unit circle, and replace condition (2) by (18), to be satisfied by $f(z)$. Then the matrix B of the two-point case is replaced by one, obtained from B by multiplying the elements of the ith row by some power of z_i. The square matrices D_r are functions of some of the z_i, and, if they are not singular, the same method can be applied as in the two-point case. Hence we have:

THEOREM 4. *Given an enumerable set of points $z_i \neq 0$, not necessarily distinct, then Theorem* 3 *holds when* (2) *is replaced by* (18), *provided that the square matrices $D_r(z_i)$ are non-singular.*

It should be noted that each of the determinants $|D_r(z_i)|$ is a mixed polynomial in some of the numbers z_i, and, if $z_i = 1$ for all i, the value of each determinant is positive. Hence those sequences $z_i \neq 0$ which make one of the determinants vanish constitute an exceptional set depending on p_i, q_j. I shall call z_i an *admissible* sequence, if none of the determinants vanishes. For a given complete pair p_i, q_j, and a given sequence z_i, the polynomials $|D_r(z_i)|$ form a *test set* of polynomials, and, when they show that a sequence is admissible, the bounds of formula (28), depending now on p_i, q_j, z_i, ensure the existence of $f(z)$. For

example in the three-point case

$$\left.\begin{aligned} z_i &= \alpha,\ 1,\ \alpha,\ 1,\ \alpha,\ 1,\ \alpha,\ 1,\ldots \\ p_i &= 0,\ 1,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ldots \\ q_j &= 0,\ 2,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ldots \end{aligned}\right\}, \tag{29}$$

$$|D_1| = \begin{vmatrix} \alpha/1! & \alpha^4/4! \\ 1 & 1/3! \end{vmatrix} = (4-\alpha^3)\alpha/4!, \qquad |D_2| = |D_3| = \ldots = 1,$$

showing that α is admissible unless it is 0 or one of the values of $\sqrt[3]{4}$.

The information about the order of magnitude required for the values of a_i, b_j can be made more precise than in Theorem 3 if we make the restriction on the sequences p_i, q_j satisfying (3) and (4) that

$$m_{r+1} - m_r \text{ is bounded.} \tag{30}$$

In such case I shall say that the complete pair is of *bounded span*. It will be of advantage to alter slightly the method of approach by treating the case symmetrically; i.e., instead of finding only the unknown coefficients of $f(z)$ at $z = 0$, to find *all* coefficients at $z = \frac{1}{2}$. This is equivalent to the following problem: *Given* $f_{p_i}(\frac{1}{2}) = a_i$, $f_{q_j}(-\frac{1}{2}) = b_j$, *to find all* x_k *of* $f(z) = \sum_{k=0}^{\infty} z^k x_k/k!$.

The corresponding simultaneous equations to be solved are

$$\sum_{k=0}^{\infty} (2^{-k}/k!)x_{k+p_i} = a_i, \qquad \sum_{k=0}^{\infty} \{(-2^{-k})/k!\}x_{k+q_j} = b_j, \tag{31}$$

written in the order of magnitude of the integers p_i, q_j, p_i preceeding q_j when they are equal. The matrix B^* of these equations is obtained from the matrix of all equations

$$\begin{bmatrix} 1 & 2^{-1}/1! & 2^{-2}/2! & 2^{-3}/3! & . & . & . \\ 1 & -2^{-1}/1! & 2^{-2}/2! & -2^{-3}/3! & . & . & . \\ 0 & 1 & 2^{-1}/1! & 2^{-2}/2! & . & . & . \\ 0 & 1 & -2^{-1}/1! & 2^{-2}/2! & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} \tag{32}$$

by retaining only rows corresponding to either a p_i or a q_j. If p_i, q_j is a complete pair, the square matrix D^*_{r+1} formed from the rows and columns m_r+1, m_r+2,..., m_{r+1}, i.e. of order $m_{r+1}-m_r$, has no zeros in its diagonal or to the right, and all elements below this matrix are zero. It cannot be singular, for the corresponding square matrix D_{r+1} is non-singular, and the non-vanishing of either determinant expresses the necessary and sufficient condition that a certain polynomial $g(u)$ of

degree $m_{r+1}-m_r$ be uniquely determined by a set of values $g_{p_i}(1)$, $g_{q_j}(0)$, $|D^*_{r+1}|$ being the determinant obtained after the substitution $u = v+\frac{1}{2}$.

Thus we can apply the method of solution by reciprocals of matrices as in Theorem 3. It follows from (30) that there are only a finite number of distinct matrices D^*_r, and, since all the rows of B^* are of one or the other pattern (this was not the case with B), the number of *distinct* functions $C^*_n(t)$, defined as in (19), is *finite*. Hence there is a fixed positive number t, the least upper bound of numbers τ satisfying both $\tau \leqslant 1$ and $C_n(\tau) \leqslant 2-\epsilon$, and we can replace (28) by

$$a_i = O(t^{p_i}), \qquad b_j = O(t^{q_j}). \tag{33}$$

Thus we have proved:

THEOREM 5. *If the pair of sequences p_i, q_j is complete and of bounded span, there exists a positive number t such that the existence of a unique integral function $f(z)$ of order one and of type t satisfying* (1) *and* (2) *is ensured when the values a_i, b_j are of magnitude* (33).

This result can now be extended to the case of a finite number of points. We consider a sequence of complex numbers $z_n \neq 0$, taking a *finite number of distinct values* only, and a sequence of integers

$$0 \leqslant p_n \leqslant p_{n+1}$$

tending to infinity, and such that $p_i \neq p_j$ when $z_i = z_j$. The sequence p_n will be called *complete and of bounded span*, if, in the notation of (3) and (4),

(i) $P(m) \geqslant m \quad (m = 1, 2, \ldots)$,

(ii) $P(m_r) = m_r$ for an infinite sequence of distinct positive integers m_r,

(iii) $m_{r+1}-m_r \leqslant K$.

Then the method, applied formerly, gives:

THEOREM 6. *If the sequence p_n is complete and of bounded span, and $z_n \neq 0$ is an admissible sequence taking a finite number of distinct values only, then there exists a positive number t, depending on the sequences p_n and z_n, such that the existence of a unique integral function $f(z)$ of order one and of type t, satisfying $f_{p_n}(z_n) = a_n$, is ensured for every sequence a_n for which*

$$a_n = O(t^n).$$

COROLLARY. *If z_n is not admissible, then $z_n+\eta$ is admissible for a continuum of arbitrary small values of η.*

To conclude, we return to example (29). Applying the altered method, we obtain:

$$|D_1^*| = \begin{vmatrix} 1 & (\alpha-\frac{1}{2})/1! & (\alpha-\frac{1}{2})^2/2! & (\alpha-\frac{1}{2})^3/3! & (\alpha-\frac{1}{2})^4/4! \\ 1 & -2^{-1}/1! & 2^{-2}/2! & -2^{-3}/3! & 2^{-4}/4! \\ 0 & 1 & 2^{-1}/1! & 2^{-2}/2! & 2^{-3}/3! \\ 0 & 0 & 1 & -2^{-1}/1! & 2^{-2}/2! \\ 0 & 0 & 0 & 1 & -2^{-1}/1! \end{vmatrix},$$

which is of the type $a_{n,k} = dP_n(u_n)/dx^k$ $(k, n = 0, 1, 2, 3, 4)$ where $u_0 = \alpha-\frac{1}{2}$, $u_1 = u_3 = u_4 = -\frac{1}{2}$, $u_2 = \frac{1}{2}$, and $P_0(x) = P_1(x) = x^4/4!$, $P_2(x) = x^3/3!$, $P_3(x) = x^2/2!$, $P_4(x) = x/1!$. This determinant is obviously invariant under translation $x+h$ since the polynomials $P_n(x)$ are all of degree less than five. For $h = \frac{1}{2}$ we have $u_0 = \alpha$, $u_1 = u_3 = u_4 = 0$, $u_2 = 1$, and the determinant is

$$\begin{vmatrix} 1 & \alpha/1! & \alpha^2/2! & \alpha^3/3! & \alpha^4/4! \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/1! & 1/2! & 1/3! \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix},$$

which reduces to $-|D_1|$.

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